

Fuzzy Transitivity and Monotonicity of Cardinality-based Similarity Measures

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Abstract The interrelationship of notions is presented in fuzzy transitivity and monotonicity of fuzzy similarity measures. It is observed that the axiom of fuzzy transitivity may replace that of monotonicity in the definition of fuzzy similarity measures.

Keywords Fuzzy similarity measures · Fuzzy transitivity · Monotonicity

1. Introduction

Since Zadeh [23] presented the definition of similarity relations, similarity measures between fuzzy sets have been widely studied and applied in various areas [6]. Turksen and Zhong [20] applied similarity measures between fuzzy sets for an approximate analogical reasoning. In a multimedia database query, Candan et al [5] applied similarity measures to develop query processing with different fuzzy semantics. There is no exact definition of similarity, though there are several context dependent definitions in literature e.g. cluster analysis [10], image classification [13], recognition [2], pattern recognition [16], rule basis [17] and diagnostics. To formulate a general purpose definition of similarity is still a challenging task. As far as the axioms for similarity are concerned, there is a general agreement on the axioms of reflexivity and symmetry, but there has always been a hesitation in including the axiom of transitivity in the postulates of similarity. Instead, the monotonicity is included as a third

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condition, although the transitivity has been very desirable property, for example, transitivity plays an important role in hierarchical clustering [21].

Beg and Ashraf have already discussed the relationship between monotonicity and weak fuzzy transitivity in [4]. They restricted their discussion to the use of min t-norm, Kleene-Dienes implicator and r -implicators. Here, the same concept will be generalized through weak transitivity for several important implicators and t-norms. The main focus will be on the Jaccard's coefficient [8] which is proved to be fuzzy transitive in case any fuzzy implicator and t-norm is used. These results present a new and more general approach to the concept of similarity measures. Moreover, our results may be helpful for hierarchical structures which require transitivity of similarity relations. The role of strong and weak fuzzy transivities will also be demonstrated.

In this paper, X represents a crisp universe of generic elements and $F(X)$ is the set of all fuzzy subsets of X . A fuzzy set A in the universe X is a mapping from X to $[0, 1]$. $A(x)$ represents the degree of membership of x in A for any $x \in X$, $A(x)$ can also be represented by x [22].

Definition 1.1 [22] *Let $A, B \in F(X)$. The containment of A into B (denoted by \subseteq) and the equality of A and B (denoted by $=$) are defined as:*

- (i) $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$.
- (ii) $A = B$ if and only if $A(x) = B(x)$ for all $x \in X$.

Triangular norms are used to model pointwise intersection of fuzzy sets.

Definition 1.2 [22] *Following fuzzy sets were defined by Zadeh for all $A, B \in F(X)$:*

- (i) $(A \cup B)(x) = \max(A(x), B(x))$ for all $x \in X$.
- (ii) $(A \cap B)(x) = \min(A(x), B(x))$ for all $x \in X$.
- (iii) $A^c(x) = 1 - A(x)$ for all $x \in X$.

Definition 1.3 [15] *A binary operation $T : [0, 1]^2 \rightarrow [0, 1]$ is a Triangular norm (t-norm) if it satisfies the following:*

- 1) $T(1, x) = x$ for all $x \in X$. (Boundary condition)
- 2) $T(x, y) = T(y, x)$ for all $x, y \in X$. (Commutativity)
- 3) $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in X$. (Associativity)
- 4) if $w \leq x$ and $y \leq z$, then $T(w, y) \leq T(x, z)$ for all $w, x, y, z \in X$. (Monotonicity)

Following are some popular choices for t-norms:

- (i) The minimum operator $M : M(x, y) = \min(x, y)$.
- (ii) The Lukasiewicz t-norm $W : W(x, y) = \max(x + y - 1, 0)$.
- (iii) The product operator $P : P(x, y) = xy$.

Definition 1.4 [11] *A fuzzy implicator I is a binary operation on $[0, 1]$ with order reversing first partial mappings and order preserving second partial mappings such that: $I(0, 1) = I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.*

Definition 1.5 [18] *Consider a left-continuous t-norm T . Then the mapping: $[0, 1]^2 \rightarrow [0, 1]$ defined by $I_T(x, y) = \sup\{z \mid z \in [0, 1] \text{ and } T(x, z) \leq y, \}$ is called an r -implicator.*

Theorem 1.1 [18] *A mapping $I : [0, 1]^2 \rightarrow [0, 1]$ is an r -implicator based on some left-continuous t -norm T iff I satisfies the following condition:*

Ordering property: $I(x, y) = 1$ iff $x \leq y$ for all $x, y \in [0, 1]$.

Remark 1.1 [19] Following next is a list of some important implicators. For all $x, y \in [0, 1]$:

(i) $I_b(x, y) = \max(1 - x, y)$;

(ii) $I_a(x, y) = \min(1 - x + y, 1)$;

(iii) $I_*(x, y) = 1 - x + xy$;

(iv) $I_{\#}(x, y) = \min(\max(1 - x, y), \max(x, 1 - x), \max(y, 1 - y))$;

(v) $I_m(x, y) = \max(1 - x, \min(x, y))$;

where, I_b is the Kleene Dienes fuzzy implicator, I_a is the Lukasiewicz fuzzy implicator and I_* is Reichenbach fuzzy implicator.

A fuzzy binary relation R from a universe X to a universe Y is a fuzzy subset of $X \times Y$. $R(x, y)$ is called the degree of relationship between x and y .

Fuzzy binary relations will be called fuzzy relations in this paper. A fuzzy relation from X to X is called a fuzzy relation on X .

Definition 1.6 [8] *Let T be a triangular norm and R a fuzzy relation on X . T is T -transitive if and only if*

$$T(R(x, y), R(y, z)) \leq R(x, z) \text{ for all } x, y, z \in X.$$

Definition 1.7 [34] *Let R be a fuzzy relation on X . The fuzzy set of transitivity $tr^{I,T}(R)$ is a fuzzy relation on X defined as:*

$$tr^{I,T}(R)(x, z) = \inf_{y \in X} I(T(R(x, y), R(y, z)), R(x, z)),$$

$tr^{I,T}(R)$ assigns a degree of transitivity to the relation at each point of $X \times X$.

If $tr^{I,T}(R) \neq \emptyset$, then the relation R is called a fuzzy transitive relation.

If $tr^{I,T}(R)(x, z) \geq 0.5$ for all $x, z \in X$, then the relation R is called a strong fuzzy transitive relation, otherwise it is called a weak fuzzy transitive relation.

Definition 1.8 [4] *The measure of fuzzy transitivity is a mapping $Tr^{I,T} : F(X \times X) \rightarrow [0, 1]$ defined as:*

$$Tr^{I,T}(R) = \inf(tr^{I,T}(R)),$$

where $F(X \times X)$ denotes the set of all fuzzy relations on X .

2. Similarity Measure for Fuzzy Sets

Definition 2.1 [12] *A mapping $S : F(X) \times F(X) \rightarrow [0, 1]$, is said to be measure of similarity between fuzzy sets if for all $A, B \in F(X)$, $S(A, B)$ satisfies the properties (P1-P3):*

(P1) $S(A, B) = 1$ if $A = B$;

(P2) $S(A, B) = S(B, A)$;

(P3) If $A, B, C \in F(X)$, such that $A \subseteq B \subseteq C$, then $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C)$.

This later condition (P3) is called *Monotonicity*.

Definition 2.2 The scalar cardinality of a fuzzy set A on a finite universe $X = \{x_1, \dots, x_n\}$ proposed by De Luca and Termini [9] is the following:

$$|A| = \sum_{x \in X} A(x).$$

Some well known cardinality based similarity measures are given in the Table 1 ([8,14]).

Table 1: Some cardinality based similarity measures.

Measure	Expression	T
Jaccard's coefficient	$\frac{ A \cap B }{ A \cup B }$	W
Simple matching	$1 - \frac{ A \Delta B }{n}$	W
Dice	$\frac{2 A \cap B }{ A \Delta B + 2 A \cap B }$	$-$
Rogers and Tanimoto	$\frac{n - A \Delta B }{n + A \Delta B }$	W
Sokal and Sneath 1	$\frac{ A \cap B }{ A \cap B + 2 A \Delta B }$	W
Sokal and Sneath 2	$1 - \frac{ A \Delta B }{2n - A \Delta B }$	$-$

Note that $|A \Delta B| = |A \setminus B| + |B \setminus A|$ and $A \setminus B = A \cap B^c$.

Theorem 2.1 Let S be a fuzzy relation. If an r -implicator is used in the calculating $Tr(S)$ and $Tr(S) < 1$, then the monotonicity (Def. 2.1 (P3)) of S will be satisfied for any left-continuous t -norm T .

Proof Let $A \subseteq B \subseteq C$ and $Tr(S) < 1$. We have to prove that $S(A, B) \geq S(A, C)$ and $S(B, C) \geq S(A, C)$.

Let on contrary suppose that either

$$S(A, B) < S(A, C) \text{ or } S(B, C) < S(A, C). \quad (1)$$

For any t -norm T ,

$$T(x, x) \leq x \text{ for all } x \in X. \quad (2)$$

(This is because $T(x, y) \leq \min(x, y)$ for all $x, y \in X$ (for detail see [1]).)

(1) and (2) together imply that:

$$T(S(A, B), S(B, C)) < S(A, C).$$

As we know $Tr(S) < 1$ implies that

$$I(T(S(A, B), S(B, C)), S(A, C)) < 1,$$

since the ordering property holds (Theorem 1.1) and I is an r -implicator so, we get $1 < 1$, which is a contradiction.

Theorem 2.2 *All the implicators mentioned in Remark 1.1 when used in the $Tr(S)$ and $Tr(S) < 0.5$, then the monotonicity of S will be satisfied for any t -norm T .*

Proof Let $A \subseteq B \subseteq C$ and $Tr(S) < 0.5$. We have to prove that $S(A, B) \geq S(A, C)$ and $S(B, C) \geq S(A, C)$.

As we know that $Tr(S) < 0.5$ implies that

$$I(T(S(A, B), S(B, C)), S(A, C)) < 0.5,$$

let $T(S(A, B), S(B, C)) = x$.

Hence

$$I(x, S(A, C)) < 0.5.$$

Case 1 If $I = I_b$,

$$\begin{aligned} I_b(x, S(A, C)) &< 0.5, \\ \max(1 - x, S(A, C)) &< 0.5, \\ \iff 1 - x &< 0.5 \text{ and } S(A, C) < 0.5, \\ \iff S(A, C) &< S(A, B) \text{ and } S(A, C) < S(B, C). \end{aligned}$$

Case 2 If $I = I_*$,

$$\begin{aligned} I_*(x, S(A, C)) &< 0.5, \\ 1 - x + xS(A, C) &< 0.5, \\ x(1 - S(A, C)) &> 0.5, \\ \iff 1 > x > 0.5 &\text{ and by using value of } x, \text{ we will get the value of } S(A, C) \\ \iff 1 > x > 0.5, &\text{ then we get } 0 < S(A, C) < 0.5, \\ \iff 0.5 < S(A, B) < 1 &\text{ and } 0 < S(A, C) < 0.5, \\ \iff S(A, C) < S(A, B) &\text{ and } S(A, C) < S(B, C). \end{aligned}$$

Case 3 If $I = I_{\#}$,

$$\begin{aligned} I_{\#}(x, S(A, C)) &< 0.5, \\ \min(\max(1 - x, S(A, C)), \max(x, 1 - x), \max(S(A, C), 1 - S(A, C))) &< 0.5. \end{aligned}$$

A possible case is only

$$\begin{aligned} \max(1 - x, S(A, C)) &< 0.5 \\ \iff 1 - x &< 0.5 \text{ and } S(A, C) < 0.5 \\ \iff x > 0.5 &\text{ and } S(A, C) < 0.5 \\ \iff x > S(A, C) \\ \iff S(A, B) > S(A, C) &\text{ and } S(A, C) < S(B, C). \end{aligned}$$

Case 4 If $I = I_m$,

$$\begin{aligned} I_m(x, S(A, C)) &< 0.5, \\ \max(1 - x, \min(x, S(A, C))) &< 0.5, \\ 1 - x &< 0.5 \text{ and } \min(x, S(A, C)) < 0.5. \end{aligned}$$

A possible case is only

$$\begin{aligned} x &> 0.5 \text{ and } S(A, C) < 0.5 \\ \iff x &> S(A, C) \\ \iff S(A, B) &> S(A, C) \text{ and } S(A, C) < S(B, C). \end{aligned}$$

Theorem 2.3 The fuzzy similarity $S(A, B) = \frac{|A \cap B|}{|A \cup B|}$ is a fuzzy transitive similarity measure for any t -norm T and for $I_\#, I_m, I_b, I_*$ and I_a implicators.

Proof Let on contrary suppose that there exist $A, B, C \in F(X)$, such that:

$$I(T(S(A, B), S(B, C)), S(A, C)) = I(T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), \frac{|A \cap C|}{|A \cup C|}) = 0.$$

Case 1 If $I = I_b$,

$$\begin{aligned} \iff \max(1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), \frac{|A \cap C|}{|A \cup C|}) &= 0 \\ \iff 1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 0 \text{ and } \frac{|A \cap C|}{|A \cup C|} &= 0 \\ \iff T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 1 \text{ and } |A \cap C| = 0 & \\ \iff \frac{|A \cap B|}{|A \cup B|} = 1, \frac{|B \cap C|}{|B \cup C|} = 1 \text{ and } |A \cap C| = 0 & \\ \iff A = B = C \text{ and } |A \cap C| = 0, & \end{aligned}$$

a contradiction.

Case 2 If $I = I_*$,

$$\begin{aligned} \iff 1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) + T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) \frac{|A \cap C|}{|A \cup C|} &= 0 \\ \iff T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) \frac{|A \cap C|}{|A \cup C|} &= 1 \\ \iff T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) (1 - \frac{|A \cap C|}{|A \cup C|}) = 1 & \\ \iff T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 1 \text{ and } \frac{|A \cap C|}{|A \cup C|} = 0 & \\ \iff \frac{|A \cap B|}{|A \cup B|} = 1, \frac{|B \cap C|}{|B \cup C|} = 1 \text{ and } |A \cap C| = 0 & \\ \iff A = B = C \text{ and } |A \cap C| = 0, & \end{aligned}$$

a contradiction.

Case 3 If $I = I_\#$,

$$\min\{\max(1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), \frac{|A \cap C|}{|A \cup C|}), \max(T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), 1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|})), \max(\frac{|A \cap C|}{|A \cup C|}, 1 - \frac{|A \cap C|}{|A \cup C|})\} = 0.$$

A possible case is only

$$\begin{aligned} &\max(1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), \frac{|A \cap C|}{|A \cup C|}) = 0 \\ \iff &1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 0 \text{ and } \frac{|A \cap C|}{|A \cup C|} = 0 \\ \iff &T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 1 \text{ and } |A \cap C| = 0 \\ \iff &\frac{|A \cap B|}{|A \cup B|} = 1, \frac{|B \cap C|}{|B \cup C|} = 1 \text{ and } |A \cap C| = 0 \\ \iff &A = B = C \text{ and } |A \cap C| = 0, \end{aligned}$$

a contradiction.

Case 4 If $I = I_m$,

$$\begin{aligned} &\max(1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), \min(T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), \frac{|A \cap C|}{|A \cup C|})) = 0, \\ &1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 0 \text{ and } \min(T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}), \frac{|A \cap C|}{|A \cup C|}) = 0 \\ \implies &T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 1 \text{ and } \{T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 0 \text{ or } \frac{|A \cap C|}{|A \cup C|} = 0\}. \end{aligned}$$

A possible case is only

$$\begin{aligned} &T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 1 \text{ and } \frac{|A \cap C|}{|A \cup C|} = 0 \\ \iff &T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 1 \text{ and } |A \cap C| = 0 \\ \iff &\frac{|A \cap B|}{|A \cup B|} = 1, \frac{|B \cap C|}{|B \cup C|} = 1 \text{ and } |A \cap C| = 0 \\ \iff &A = B = C \text{ and } |A \cap C| = 0, \end{aligned}$$

a contradiction.

Case 5 If $I = I_a$,

$$\begin{aligned} \iff &1 - T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) + \frac{|A \cap C|}{|A \cup C|} = 0 \\ \iff &T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) - \frac{|A \cap C|}{|A \cup C|} = 1 \\ \iff &T(\frac{|A \cap B|}{|A \cup B|}, \frac{|B \cap C|}{|B \cup C|}) = 1 \text{ and } \frac{|A \cap C|}{|A \cup C|} = 0 \\ \iff &\frac{|A \cap B|}{|A \cup B|} = 1, \frac{|B \cap C|}{|B \cup C|} = 1 \text{ and } |A \cap C| = 0 \\ \iff &A = B = C \text{ and } |A \cap C| = 0, \end{aligned}$$

a contradiction.

Corollary 2.1 *The fuzzy similarity*

$$S(A, B) = 1 - \frac{|A\Delta B|}{n}$$

is a fuzzy transitive similarity measure for any t -norm T and for $I_{\#}, I_m, I_b, I_*$ and I_a implicators.

Corollary 2.2 *So is the fuzzy similarity*

$$S(A, B) = \frac{2|A \cap B|}{|A\Delta B| + 2|A \cap B|}.$$

Corollary 2.3 *The fuzzy similarity*

$$S(A, B) = \frac{n - |A\Delta B|}{n + |A\Delta B|}$$

is the same measure.

Corollary 2.4 *The fuzzy similarity*

$$S(A, B) = \frac{|A \cap B|}{|A \cap B| + 2|A\Delta B|}$$

is the similar measure.

Corollary 2.5 *So is the fuzzy similarity*

$$S(A, B) = 1 - \frac{|A\Delta B|}{2n - |A\Delta B|}.$$

3. Conclusion

In this paper, it is concluded that the weak transitive fuzzy relations satisfy monotonicity. This result is proved for all the implicators $I_b, I_*, I_{\#}$ and I_m and for any t -norm T . Moreover it proves not only that the Jaccard's coefficient is fuzzy transitive relation for $I_b, I_*, I_a, I_{\#}$ and I_m and for any t -norms, but also that the simple matching, dice, Rogers and Tanimoto, Sokal and Sneath (1 and 2) are fuzzy transitive relation for $I_b, I_*, I_a, I_{\#}$ and I_m and for any t -norms T .

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