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Convex hesitant fuzzy sets

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Abstract. Convex hesitant fuzzy sets are define as an extension of convex fuzzy sets. Also level sets are defined for hesitant fuzzy sets and discussed with their convexity. We focus on aggregation functions for hesitant fuzzy elements. These aggregation functions are further extended for hesitant fuzzy sets as well as for the convex structures of these sets.

Keywords: Fuzzy set, hesitant fuzzy set, convexity, aggregation function

1. Introduction

Different kinds of convexity have been introduced and used in the literature for vector space universe [2, 3, 19, 20]. These definitions allow us to deal with any possible kind of crisp sets, fuzzy sets or intuitionistic fuzzy set. Several attempts are made to give suitable notion of convexity for cases when the universe is not necessarily a vector space. Looking for some general definition of this kind, the study [10] comes into our rescue. In his seminal paper Llinares [10], gave several generalizations of the concept of a convex crisp set and analyzed it in order to deal with fixed point theory, in the search for generalizations of classical theorems (e.g.: Brouwer's fixed point theorem, Knaster-Kuratowski-Mazurkiewicz theorem, etc). In this paper we introduce another generalized definition, valid in a much wider setting, which is based on the ideas of $k-$convex structure and equiconnected space given in [10, Definition 4 and 5].

Let X be a non empty set. A convex structure on X is a map $H : X \times X \times [0, 1] \to X$ that satisfies the following properties:

- $H(x, y, \lambda) = H(y, x, 1 - \lambda)$, for every $x, y \in X$ and $\lambda \in [0, 1]$.
- $H(x, x, \lambda) = x$, for every $x \in X$ and $\lambda \in [0, 1]$.
- $H(x, y, 1) = x$, for every $x, y \in X$.

A crisp subset A of X is said to be convex with respect to H, or $H-$convex for short, if $H(x, y, \lambda) \in A$, for all $x, y \in A$ and for all $\lambda \in [0, 1]$. Thus, the concept of $H-$convex sets is a generalization of the classical concept of convex set [7].

Hesitant fuzzy sets are used to handle the multicriteria group decision making problems in [4, 14, 16, 17]. Wei et al. [15] discussed distance and similarity measures for hesitant interval valued fuzzy sets. We devote our study for the convexity of hesitant fuzzy sets because convexity is one of the important aspects in the study of geometric properties of hesitant fuzzy sets, mainly arising and indeed playing a crucial role in applications connected to optimization and control [1, 6]. In particular we are interested in a more general framework, by using aggregation functions instead of the particular case of t-norms (for detail of t-norms, see [9]). Accordingly, the use of aggregation functions for hesitant fuzzy sets immediately carries the question of characterizing those aggregation functions that preserve convexity in some way.

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Zadeh [19] introduced the convexity of fuzzy sets as: A fuzzy set A of the universe X is said to be convex, if for all $x, y \in X$ and $\lambda \in [0, 1]$ it holds that

$$A(\lambda x + (1 - \lambda)y) \geq \lambda A(x) + (1 - \lambda)A(y),$$

where $A(x)$ denotes the degree of membership of x in A. Zadeh's definition of convexity has at least two weak points: The first one is that it is not suitable for the case of a lattice valued fuzzy set, as the addition in the lattice is, in general, not defined. And the second one is that under this definition a fuzzy set for which all its level sets are convex, need not be convex. Thus, when maximizing a fuzzy decision (see [5] for details), fuzzy convexity in [19] does not ensure that a local maximizer is a global maximizer. This leads to a more restrictive concept of fuzzy convexity as in [1], which ensures that a local maximizer is also a global maximizer as shown in [11]. According to Ammar and Metz [1]: A fuzzy set A of the universe X is said to be quasi-convex if for all $x, y \in X$, $\lambda \in [0, 1]$ it holds that

$$A(\lambda x + (1 - \lambda)y) \geq \min\{A(x), A(y)\}.$$

In [1] it is shown that it overcomes above mentioned problem. As a matter of fact, it can also be used for mappings into partially ordered vector spaces and the class of all quasi-convex fuzzy sets is exactly the class of those fuzzy sets, for which all their $\alpha-$cuts are convex. Whenever the universe X fails to be a vector space, new definitions are needed.

The structure of the paper goes as follows: In Section 2, we recall basic notions of hesitant fuzzy set, score and $\alpha-$cut for hesitant fuzzy sets. In Section 3, we introduce the convex hesitant fuzzy set and quasi convex hesitant fuzzy set with example and establish some equivalent definitions. Relation between convex hesitant fuzzy set and quasi convex hesitant fuzzy set is also discussed. In Section 4, aggregation functions are proposed for hesitant fuzzy elements and hesitant fuzzy sets to preserve the convexity in hesitant fuzzy set. In Section 5, applications of convex hesitant fuzzy sets, based on aggregation function is given. An example to illustrate it is also constructed.

2. Hesitant fuzzy sets

Torra [13] introduced an extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set. Hesitant fuzzy set (HFS) is defined in terms of a function that returns a set of membership values for each element in the domain.

Definition 2.1. [13] A hesitant fuzzy set A on X is a function h^A that when applied to X returns a finite subset of [0, 1], which can be represented as the following mathematical symbol:

$$A = \{(x,\ h^A(x))|x \in X\},$$

where $h^A(x)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set A. For convenience, Xia and Xu [18] named $h^A(x)$ a hesitant fuzzy element (HFE).

The support of HFS A is the crisp set $Supp(A) = \{t \in X : \max h^A(t) \neq 0\} \subseteq X$, whereas the kernel of HFS A is the crisp set $Ker(A) = \{t \in X : \max h^A(t) = 1\} \subseteq X$. The HFS A is said to be normal provided that it has non empty kernel.

Definition 2.2. [13] For a hesitant fuzzy set represented by its membership function h, we define its complement as follows:

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}.$$

It is noted that the number of values in different HFEs may be different. Let $l_{h^A(x)}$ be the number of values in $h^A(x)$.

Definition 2.3. [18] For a HFE h, $s(h) = \frac{1}{l_h} \sum_{\gamma \in h} \gamma$, is called the score function of h, where l_h is the number of the elements in h and $s(h) \in [0, 1]$. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_2 \prec h_1$, if $s(h_1) = s(h_2)$, then $h_1 \approx h_2$.

Motivated by this comparison between HFEs, we can define the subset relation between HFSs.

Definition 2.4. Let A_1, A_2 be two HFSs on X.

1. If $s(h^{A_1}(x)) \leq s(h^{A_2}(x))$ for all $x \in X$ then $A_1 \subseteq A_2$.
2. If $s(h^{A_1}(x)) = s(h^{A_2}(x))$ for all $x \in X$ then $A_1 \doteqdot A_2$.

Xia and Xu [18] define some operations on the HFEs h, h_1 and h_2 :

1. $kh = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^k\}$;
2. $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;

In fact, all the above operations on HFEs are suitable for HFS [18].

Lemma 2.5. *Let h, h_1 and h_2 be three hesitant fuzzy elements. Then*

i. $s(h) = 1 - s(h^c)$.
ii. If $s(h_1) \leq s(h_2)$ then $s(h_1^c) \geq s(h_2^c)$.

Proof.

i.

$$s(h) = \frac{1}{l_h} \sum_{\gamma \in h} \gamma$$
$$= 1 - 1 + \frac{1}{l_h} \sum_{\gamma \in h} \gamma$$
$$= 1 - \left(1 - \frac{1}{l_h} \sum_{\gamma \in h} \gamma\right)$$
$$1 - \frac{1}{l_h} \sum_{\gamma \in h} (1 - \gamma)$$
$$= 1 - s(h^c)$$

ii. Easily follows by using part (i). ■

Definition 2.6. Given $\alpha \in (0, 1]$, the crisp subset of X defined by $h_\alpha^A = \{t \in X : s(h^A(t)) \geq \alpha\}$ is said to be the $\alpha-$cut (level set) of the HFS A.

3. Convex Hesitant Fuzzy Sets

Thus, following the concept of convexity for crisp sets we may introduce in a natural way the following definition of a convex hesitant fuzzy set (CHFS).

Definition 3.1. Let X be a vector space. A hesitant fuzzy set A of the universe X is said to be convex, if for all x, $y \in X$ and $\lambda \in [0, 1]$ it holds that $s(h^A(\lambda x + (1 - \lambda)y)) \geq s(\lambda h^A(x) \oplus (1 - \lambda)h^A(y))$.

Definition 3.1 has two weak points. The first one is that all the level sets of hesitant fuzzy set A are convex but A need not to be CHFS. The second one is that it is not suitable for the case, X is not a vector space, as the addition and scalar multiplication is not defined. Example 3.2 describe the first weak point of Definition 3.1 in detail.

Example 3.2. Let X be the real line. Let A be the HFS of X given by $h^A(x) = \{0.2, 0.25, 0.27\}$ if $x \leq 0$ and $h^A(x) = \{0.3, 0.35, 0.4\}$ otherwise. The level sets are obviously convex sets of the real line. However, the HFS A fails to be convex: If $\lambda = 0.5$, $x = -5$ and $y = 3$ we may observe that $0.5x + (1 - 0.5)y = -1$, so that $h^A(0.5x + (1 - 0.5)y) = \{0.2, 0.25, 0.27\}$ and $s(h^A(0.5x + (1 - 0.5)y)) = s(\{0.2, 0.25, 0.27\}) = 0.24$ whereas $0.5(h^A(x)) = \{0.1055, 0.1339, 0.1455\}$ and $(1 - 0.5)h^A(y) = \{0.1633, 0.1937, 0.2254\}$, $0.5(h^A(x)) \oplus (1 - 0.5)h^A(y) = \{0.2516, 0.2788, 0.307, 0.275, 0.3017, 0.329, 0.285, 0.311, 0.338\}$. So $s(\lambda h^A(x) \oplus (1 - \lambda)h^A(y)) = 0.297$.Thus A is not CHFS.

Thus we modify the definition of convexity for HFS.

Definition 3.3. Let X be a vector space. A HFS A of the universe X is said to be quasi-CHFS if for all x, $y \in X$, $\lambda \in [0, 1]$ it holds that $s(h^A(\lambda x + (1 - \lambda)y)) \geq \min\{s(h^A(x)), s(h^A(y))\}$.

Proposition 3.4. *Let X be a universe. The following statements are equivalent:*

1. *A is a quasi-CHFS,*
2. *Any $\alpha-$cut of HFS A is convex crisp set.*

Proof. Let us consider $\alpha \in (0, 1]$ and A be quasi-CHFS. If x, $y \in (h^A)_\alpha$ then $s(h^A(\lambda x + (1 - \lambda)y)) \geq \min\{s(h^A(x)), s(h^A(y))\} \geq \alpha$.

Therefore $\lambda x + (1 - \lambda)y \in (h^A)_\alpha$, that is, $(h^A)_\alpha$ is a convex crisp set.

Conversely, if A is a HFS, whose $\alpha-$cuts are convex crisp sets for any α, we can consider the case $\alpha = \min\{s(h^A(x)), s(h^A(y))\}$ for any $(x, y) \in X \times X$. Thus, x, $y \in (h^A)_\alpha$. Thus $\lambda x + (1 - \lambda)y \in (h^A)_\alpha$. This means that $s(h^A(\lambda x + (1 - \lambda)y)) \geq \alpha = \min\{s(h^A(x)), s(h^A(y))\}$. ■

Theorem 3.5. *Let X be a vector space. If A is CHFS then A is a quasi-CHFS.*

Proof. Let A be a CHFS then $s(h^A(\lambda x + (1 - \lambda)y)) \geq s(\lambda h^A(x) \oplus (1 - \lambda)h^A(y))$ for all x, $y \in X$, $\lambda \in [0, 1]$.

This implies that $s(\lambda h^A(x) \oplus (1 - \lambda)h^A(y)) \geq s(h^A(x))$ or $s(\lambda h^A(x) \oplus (1 - \lambda)h^A(y)) \geq s(h^A(y))$ for all x, $y \in X$, $\lambda \in [0, 1]$.

This further implies that $s(h^A(\lambda x + (1 - \lambda)y)) \geq \min\{s(h^A(x)), s(h^A(y))\}$.

Hence A is a quasi-CHFS. ■

Example 3.6. A quasi-CHFS A may not be CHFS: The HFS A given in Example 3.2 is not CHFS. As the level sets are obviously convex sets of the real line. So by Proposition 3.4 it is easy to say that A is a quasi-CHFS.

Whenever the universe X fails to be a vector space, we need to improve the Definition 3.1 This is done as follows:

Definition 3.7. Let X be a universe which may not be a vector space. Assume that X is equipped with a convex structure through a map $H : X \times X \times [0, 1] \rightarrow X$ (given in Section 1). Then, a HFS A of X is said to be an $H-$CHFS if for every $x, y \in X$ and every $\lambda \in [0, 1]$ it holds that $s(h^A(H(x, y, \lambda))) \geq \min\{s(h^A(x)), s(h^A(y))\}$.

Proposition 3.8. *Let X be a universe and let H be a convex structure on X. Then the following statements are equivalent:*

1. *A is an $H-$CHFS.*
2. *Any $\alpha-$cut of HFS A is an $H-$convex crisp set.*

Proof. Let us consider $\alpha \in (0, 1]$ and an $H-$CHFS A. If $x, y \in (h^A)_\alpha$, then $s(h^A(H(x, y, \lambda))) \geq \min\{s(h^A(x)), s(\min h^A(y))\} \geq \alpha$. And therefore, $H(x, y, \lambda) \in (h^A)_\alpha$, that is, $(h^A)_\alpha$ is an $H-$convex crisp set.

Conversely, if A is a HFS, whose $\alpha-$cuts are $H-$convex crisp sets for any α, we can consider the case $\alpha = \min\{s(h^A(x)), s(h^A(y))\}$ for any $(x, y) \in X \times X$. Thus, $x, y \in (h^A)_\alpha$ and then $H(x, y, \lambda) \in (h^A)_\alpha$. This means that $s(h^A(H(x, y, \lambda))) \geq \alpha = \min\{s(h^A(x)), s(h^A(y))\}$. Therefore A is an $H-$CHFS. ■

Since quasi-CHFS implies that any $\alpha-$cut is a convex crisp set. This is equivalent to the $H-$convexity with respect to $H(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Quasi-convexity implies a particular case of $H-$convexity in the HFS case. The converse is not true in general [7, Remark 9].

4. Aggregation function for convex hesitant fuzzy sets

One of the properties that make convexity important is its preservation under intersections. In this section we study this question for the case of CHFS. However, instead of the intersection we will consider a wider class of aggregation functions. Aggregation functions are usually defined on the interval [0, 1], but in our framework they have to be defined in a particular subset of [0, 1]. More interesting is that we proposed the aggregation function from the class of all HFEs to HFE. Xia and Xu [18] defined several aggregation functions for HFEs. But here we proposed aggregation function with detailed study of their properties specially design to handle the class of CHFSs.

Definition 4.1. Let E be a collection of all HFEs on X, T is an aggregation function on E, $T : E^n \rightarrow E$ which satisfies the following properties:

- Boundary Condition: $T(\{0\}, \{0\}, ..., \{0\}) = \{0\}$ and $T(\{1\}, \{1\}, ..., \{1\}) = \{1\}$
- Monotonicity: If $h_i \preceq h'_i$ then $T(h_1, h_2, ..., h_n) \preceq T(h'_1, h'_2, ..., h'_n)$
- Commutativity: For every permutation σ of $\{1, 2, \ldots, n\}$ satisfies $T(h_1, h_2, ..., h_n) \approx T(h_{\sigma(1)}, h_{\sigma(2)}, ..., h_{\sigma(n)})$
- Idempotence: $T(h, h, ..., h) \approx h$
- Compensation: $\min_{i-1}^{n}(s(h_i)) \preceq T(h_1, h_2, ..., h_n) \preceq \max_{i-1}^{n}(s(h_i))$

Proposition 4.2. *Let E be a collection of all HFEs on X. Then T is an aggregation function on E, $T : E^n \rightarrow E$ such that $T(h_1, h_2, ..., h_n) = \{s(h_i)$ for all $i\}$.*

Proof. Let $h_1, h_2, ..., h_n$ be HFEs. Then $s(h_i) \in [0, 1]$ for $1 \leq i \leq n$.

Construct a set $\{s(h_i)$ for $1 \leq i \leq n\}$ will be finite subset of [0, 1]. Therefore, $\{s(h_i)$ for $1 \leq i \leq n\}$ will be a HFE. Hence $T(h_1, h_2, ..., h_n) \in E$.

- *Boundary Condition:* Since $s(\{0\}) = 0$ then by definition of T we get $T(\{0\}, \{0\}, ..., \{0\}) = \{0\}$. Similarly $s(\{1\}) = \{1\}$ then again by definition of T we get $T(\{1\}, \{1\}, ..., \{1\}) = \{1\}$.
- *Monotonicity:* Consider $h_i \preceq h'_i$ then it means that $s(h_i) \leq s(h'_i)$. Since $T(h_1, h_2, ..., h_n) = \{s(h_i)$ for all $i\}$. Therefore, $s(T(h_1, h_2, ..., h_n)) \leq s(T(h'_1, h'_2, ..., h'_n))$.
- *Commutativity:* Since $s(h) = \frac{1}{l_h} \sum_{\gamma \in h} \gamma$ for a HFE h. Then $s(h)$ does not depend on the position of γ in h. Similarly $s(T(h_1, h_2, ..., h_n)) = s(T(h_{\sigma(1)}, h_{\sigma(2)}, ..., h_{\sigma(n)}))$. Therefore, $T(h_1, h_2, ..., h_n) \approx T(h_{\sigma(1)}, h_{\sigma(2)}, ..., h_{\sigma(n)})$.
- *Idempotence:* By definition of T we can write that $T(h, h, ..., h) = \{s(h)\}$. Therefore, $s(T(h, h, ..., h)) = s(h)$. Hence $T(h, h, ..., h) \approx h$.

– *Compensation:* Since $s(h) = \frac{1}{l_h}\sum_{\gamma\in h}\gamma$ for a HFE h. Then it is easy to note that $\min(h) \le s(h) \le \max(h)$. As we know that $T(h_1, h_2, ..., h_n) = \{s(h_i) \text{ for all } i\}$ is a HFE. So $\min_{i-1}^{n}(s(h_i)) \le s(\{s(h_i) \text{ for all } i\}) \le \max_{i-1}^{n}(s(h_i))$. Hence $\min_{i-1}^{n}(s(h_i)) \preceq T(h_1, h_2, ..., h_n) \preceq \max_{i-1}^{n}(s(h_i))$. ■

Now we are in a position to define an aggregation function for HFSs.

Definition 4.3. Let $\tilde{E}$ be a collection of all HFSs on X, $\widetilde{T}$ is an aggregation function on $\tilde{E}$, $\widetilde{T} : \tilde{E}^n \to \tilde{E}$ which satisfies the following properties:

– Boundary Condition: $\widetilde{T}(\emptyset, \emptyset, ..., \emptyset) = \emptyset$ and $\widetilde{T}(X, X, ..., X) = X$
– Monotonic: If $A_i \subseteq B_i$ then $\widetilde{T}(A_1, A_2, ..., A_n) \subseteq \widetilde{T}(B_1, B_2, ..., B_n)$
– Commutative: For every permutation σ of $\{1, 2, \ldots, n\}$ satisfies $\widetilde{T}(A_1, A_2, ..., A_n) \approx \widetilde{T}(A_{\sigma(1)}, A_{\sigma(2)}, ..., A_{\sigma(n)})(x)$
– Idempotent: $\widetilde{T}(A, A, ..., A) \approx A$
– Compensation:
$$\left\{\left\langle\left\{\min_{i-1}^{n}(s(h^{A_i}(x)))\right\}\right\rangle \mid x \in X\right\} \subseteq \widetilde{T}(A_1, A_2, ..., A_n) \subseteq \left\{\left\langle\left\{\max_{i-1}^{n}(s(h^{A_i}(x)))\right\}\right\rangle \mid x \in X\right\}$$

Proposition 4.4. *Let $\tilde{E}$ be a collection of all HFSs on X. Then $\widetilde{T}$ is an aggregation function on $\widetilde{E}$, $\widetilde{T} : \widetilde{E}^n \to \widetilde{E}$ such that*
$$\widetilde{T}(A_1, A_2, ..., A_n) = \left\{\left\langle\{T(h^{A_1}(x), h^{A_2}(x), ..., h^{A_n}(x))\}\right\rangle | x \in X\right\}.$$

Proof. It is easy to see that $\widetilde{T}$ is depend on T. So by proposition 4.2 it can be concluded that $\widetilde{T}$ satisfying definition 4.3. ■

Score of HFE plays an important role in the concept of convexity for HFS and in the aggregation functions for HFEs as well as HFSs. The construction of aggregation functions to preserve the quasi-convexity of fuzzy sets are given by Janiš et al. in [8]. So it is important to discuss that can we define aggregation function on the class of CHFS?

Proposition 4.5. *Let $\hat{E}$ be a collection of all CHFSs on X. Then $\widetilde{T}$ is an aggregation function on $\hat{E}$, $\widetilde{T} : \hat{E}^n \to \hat{E}$ such that*
$$\widetilde{T}(A_1, A_2, ..., A_n) = \left\{\left\langle\{T(h^{A_1}(x), h^{A_2}(x), ..., h^{A_n}(x))\}\right\rangle \mid x \in X\right\}.$$

Proof. Because of proposition 4.4, here it is enough to show that if $A_1, A_2, ..., A_n$ are CHFSs then $\widetilde{T}(A_1, A_2, ..., A_n)$ is also CHFS.

Let $A_1, A_2, ..., A_n$ be CHFSs then by Definition 3.1, we have $s(h^{A_i}(\lambda x + (1-\lambda)y)) \ge s(\lambda h^{A_i}(x) + (1-\lambda) h^{A_i}(y))$ for all $x, y \in X$, $\lambda \in [0, 1]$ and $1 \le i \le n$.

By the monotonicity of T, we have $T(h^{A_1}(\lambda x + (1-\lambda)y), h^{A_2}(\lambda x + (1-\lambda)y), ..., h^{A_n}(\lambda x + (1-\lambda)y)) \succeq T((\lambda h^{A_1}(x) + (1-\lambda)h^{A_1}(y)), (\lambda h^{A_2}(x) + (1-\lambda)h^{A_2}(y)), ..., (\lambda h^{A_n}(x) + (1-\lambda)h^{A_n}(y)))$ for all $x, y \in X$ and $\lambda \in [0, 1]$.

As we know that $\widetilde{T}$ is monotone and depend on T. Therefore, $s(h^{\widetilde{T}(A_1,A_2,...,A_n)}(\lambda x + (1-\lambda)y)) \ge s(\lambda h^{\widetilde{T}(A_1,A_2,...,A_n)}(x) + (1-\lambda)h^{\widetilde{T}(A_1,A_2,...,A_n)}(y))$ for all $x, y \in X$ and $\lambda \in [0, 1]$.

Hence $\widetilde{T}(A_1, A_2, ..., A_n)$ is CHFS. ■

Remark 4.6. According to Proposition 4.5 we can formulate $\widetilde{T}$ from the class of all quasi-CHFSs (resp. the class of all H-CHFSs) to quasi-CHFS (resp. H-CHFS).

Motivated by Syau and Lee [12], the above CHFS and aggregation results can be used to optimize fuzzy problems under different conditions. These operators can be used to aggregate the resulting quasi-CHFS goals, quasi-CHFS constraints, and the resulting fuzzy decision is also a quasi-CHFS.

5. Application of CHFS in optimization

CHFSs and their aggregation function can be used to optimize the decision making problems under different conditions. Here we will apply the aggregation function for CHFSs in general. A collection of HFSs $A_1, A_2, ..., A_m$ for m criteria, defined on the decision space X. An optimized decision on X is defined by the hesitant membership value from the aggregated CHFS of HFSs $A_1, A_2, ..., A_m$. We use $\widetilde{T}$ as an appropriate aggregation operator. If there exists a subset of X in which $\mu_{\widetilde{T}(A_1,A_2,...,A_m)}(x)$ reaches its maximum, then that subset of X is called the set of maximizing decisions. An element from X is said to be an efficient solution to the multiple objectives, $A_1, A_2, ..., A_m$, if no one objective can be improved without a simultaneous detriment to at least one of the other objectives. An efficient solution is also known as non-dominated solution.

Now, we discuss a numerical example for optimization problem based on convexity of HFSs. Let us consider a universal set $X = \{Diana, Adam, Bob, John\}$ and map $H : X \times X \times [0, 1] \to X$ defined by

$$H(x, y, \lambda) = \begin{cases} x & \text{if} \quad \lambda = 1 \\ y & \text{if} \quad \lambda = 0 \\ x & \text{if} \quad \lambda \in (0, 1) \text{ and } x = y \\ Diana & \text{if} \quad \lambda \in (0, 1) \text{ and } x \neq y \end{cases}$$

$Diana \in X$ is a trivial element to define a map for convex structure on X. $A = \{(Diana, \{1\}), (Adam, \{0.4, 0.6\}), (Bob, \{0.5, 0.6\}), (John, \{0.7\})\}$ is a HFS of hard working persons on X and $B = \{(Diana, \{1\}), (Adam, \{0\}), (Bob, \{0.6\}), (John, \{0.2, 0.1\})\}$ is a HFS of intelligent persons on X. HFSs A and B are $H-$CHFS and the non-trivial optimize solution from A and B is $John$ and Bob, respectively. Now by using aggregation function $\widetilde{T}$, we can aggregate HFSs A and B.

$\widetilde{T}(A, B) = \{(Diana, \{1\}), (Adam, \{0, 0.5\}), (Bob, \{0.55, 0.6\}), (John, \{0.15, 0.7\})\}$.

As an optimization problem, we are interested to find the smallest non-trivial crisp convex subset of X against the greatest value of α. For $\alpha = 0.575$, a non-trivial optimized solution is Bob from $\widetilde{T}(A, B)$.

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