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# Impact of extended Starobinsky model on evolution of anisotropic, vorticity-free axially symmetric sources

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**Abstract.** We study the implications of $R^n$ extension of Starobinsky model on dynamical instability of Vorticity-free axially symmetric gravitating body. The matter distribution is considered to be anisotropic for which modified field equations are formed in context of $f(R)$ gravity. In order to achieve the collapse equation, we make use of the dynamical equations, extracted from linearly perturbed contracted Bianchi identities. The collapse equation carries adiabatic index $\Gamma$ in terms of usual and dark source components, defining the range of stability/insatbility in Newtonian (N) and post-Newtonian (pN) eras. It is found that supersymmetric supergravity $f(R)$ model represents the more practical substitute of higher order curvature corrections.

**Keywords:** modified gravity, massive stars, dark energy theory

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## Contents



## 1 Introduction

The compact gravitating objects are commonly studied by assuming spherical symmetry. The observational signatures suggests that self gravitating compact sources may deviate from spherical symmetry. These incident deviations from spherical symmetry provides scope for the study of axially symmetric gravitating systems. However, such line element in accordance with Weyl gauge largely constrains the class of possible sources (static/dynamic) [1]. Although the consideration of non-static sources and angular momentum leave complicated analysis, but lack of the spherical symmetry in most realistic scenario must be considered. Thus, the explorations regarding outcome of stellar evolution with assumption of axial symmetry is worthwhile.

The analysis of stability range plays significant role in relevant fields, such as astronomy, astrophysics and structure formation theories. The gravitating bodies remain stable as long as the balance is being maintained between outward drawn pressure induced by internal fusion and the inward gravitational pull [2]. The stellar collapse originates from the situations where gravity dominates as a consequence of internal fuel consumption. Instability range of compact sources varies along with their mass, supermassive stars has tendency of wider instability range implying smaller life span [3–5], while stars having mass of the order of one solar mass tends to be more stable.

The initial contribution on dynamical instability was of Chandrasekhar [6], he described a pattern for the representation of instability range in terms of adiabatic index for ideal matter distribution. Hillebrandt and Steinmetz [7] found the instability criterion for compact objects having pressure anisotropies in matter configuration. Many authors [8–18], emphasized on stellar evolution and stability range for a number of matter distributions, such as isotopic fluid, anisotropy, zero expansion, shear-free condition, radiation and dissipation. The results obtained from their analysis established a major remark that nominal variations in fluid configuration alters stability range significantly. Also, the high radiation transport leaves instabilities in the system.

Modified theories have gained more attention to count with the issue of cosmic acceleration [19–28]. Likewise in general relativity (GR), the instability problem has also been

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widely discussed in modified gravity theories namely, $f(R)$, $f(R,T)$, where $T$ is the trace of energy momentum tensor, $f(G)$, Brans-Dicke theory etc. The modified theories provide higher order corrections to GR on large scale structures for inclusion of dark energy substitutes. People [29–35], worked out the instability problem in $f(R)$ theory for various matter distributions with and without Maxwell source, concluding that the inclusion of higher order curvature terms depicts the more broader picture of factors affecting the stability. The evolution of compact bodies in Gauss-Bonnet ($f(G)$) and $f(\mathcal{T})$ (where $\mathcal{T}$ is the torsion scalar) theory has been explored in [36–38]. Recently, the dynamics of isotropic and anisotropic fluid has been studied in $f(R,T)$ theory [39, 40].

$f(R)$ gravity represents the elementary modification in GR, including higher order curvature terms in Einstein-Hilbert (EH) action. In $f(R)$, the Ricci scalar $R$ modifies to $\sqrt{-g}f(R)$ in EH action incorporating higher curvature terms [41–44], when $g$ is metric tensor and $f(R)$ stands for the general function of $R$. The EH action in $f(R)$ is of the form

$$S_{f(R)} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S\,, \tag{1.1}$$

where $\kappa$ stands for the coupling constant and $S$ is the action for usual matter. Any specific form of $f(R)$ is viable, if it is in accordance with the viability criterion, i.e., the second order derivative of considered model must be positive along with realistic situations such as weak lensing, cosmic microwave background and clustering spectrum [45–48].

In GR, the anisotropic dissipative and shear-free fluid models have been studied for the dynamic, axially and reflection symmetric sources [49, 50]. In this study authors ignore the rotation about symmetric axis i.e., $dtd\phi$ term is absent in general axial symmetry. Sharif and Zousaf [51] explained the dynamics of restricted non-static axially symmetric collapse for Starobinsky model [52] in anisotropic environment. Herein, we are aiming to discuss the impact of supersymmetric Starobinsky model constructed in [53] on dynamics of the axially symmetric gravitating source. The components of modified field equations are used to construct the conservation equations for anisotropic matter configuration. We have implemented the perturbation approach in order to develop the collapse equation and discuss the role of usual matter and dark source components in establishment of stabilit/instability range. Perturbation of dynamical equations lead to the collapse equation which is further used to discuss instability range in terms of $\Gamma$ for N and pN regimes.

The manuscript arrangement is: section 2 contains the components of field equations and conservation equations for axially symmetric self gravitating objects along with the description of anisotropic matter configuration. The $f(R)$ model is furnished in section 3 along with the perturbed field equations and Bianchi identities. The collapse equation and dynamical analysis in N and pN eras is provided in section 4. The last section constitutes the conclusion followed by an appendix.

## 2 Dynamical equations

For the dynamical analysis, we consider the spacetime which describes the restricted non-static axial symmetry avoiding the terms of reflection and rotation about the the symmetry axis. The reduced form of general axially symmetric spacetime in spherical coordinates is [54]

$$ds^2 = -A^2(t,r,\theta)dt^2 + B^2(t,r,\theta)dr^2 + B^2(t,r,\theta)r^2d\theta^2 + C^2(t,r,\theta)d\phi^2. \tag{2.1}$$

It is worthwhile to mention here that, this analysis belongs to the restricted class of axially symmetric sources, i.e., absence of meridional motions and vorticity. In general axial symmetry, five independent metric functions should appear in the line element, due to restricted character (excluding meridional motions and motions around symmetry axis), we have three of them. In fact,we are dealing with analytic approach to present the dynamical analysis of gravitating source containing three independent metric functions. Already, it is a cumbersome task to analyze the system without reflection and rotation then rather to include such terms so we have neglected the terms $dtd\theta$ and $dtd\phi$ in the general axially symmetric line element.

The gravitating source is considered to have anisotropic matter configuration defined by the energy momentum tensor [1]

$$T_{uv} = (\rho + p_\perp)V_uV_v - \left(K_uK_v - \frac{1}{3}h_{uv}\right)(P_{zz} - P_{xx}) - \left(L_uL_v - \frac{1}{3}h_{uv}\right)(P_{zz} - P_{xx}) + Pg_{uv} + 2K_{(u}L_{v)}P_{xy}\,, \tag{2.2}$$

where

$$P = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz}), \quad h_{uv} = g_{uv} + V_uV_v\,,$$

$P_{xx}, P_{yy}, P_{zz}$ and $P_{xy}$ denote different stresses inducing pressure anisotropy, provided that $P_{xy} = P_{yx}$ and $P_{xx} \neq P_{yy} \neq P_{zz}$. The energy density is labeled as $\rho$, $V_u$ is for four-velocity, $K_u$ and $L_u$ denote four vectors in radial and axial directions respectively. We have chosen Eulerian frame to describe the quantities, implying that

$$V_u = -A\delta^0_u\,, \quad K_u = B\delta^1_u1\,, \quad L_u = rB\delta^0_u\,. \tag{2.3}$$

The variation of EH action (1.1) with respect to metric tensor $g_{uv}$ yields the following field equations [35]

$$f_RR_{uv} - \frac{1}{2}f(R)g_{uv} - \nabla_u\nabla_vf_R + g_{uv}\Box f_R = \kappa T_{uv}\,, \quad (u, v = 0, 1, 2, 3)\,, \tag{2.4}$$

where $\nabla_u$ is covariant derivative, $\Box = \nabla^u\nabla_v$, $f_R \equiv df(R)/dR$. Likewise GR, we may write field equations as

$$G_{uv} = \frac{\kappa}{f_R}[\overset{(D)}{T}_{uv} + T_{uv}] = \frac{\kappa}{f_R}\overset{(D)}{T}_{uv} + \frac{\kappa}{f_R}T_{uv}\,, \tag{2.5}$$

where $\overset{(D)}{T}_{uv}$ represents energy momentum tensor of dark source contribution given by

$$\overset{(D)}{T}_{uv} = \frac{1}{\kappa}\left[\frac{f(R) - Rf_R}{2}g_{uv} + \nabla_u\nabla_vf_R - g_{uv}\Box f_R\right]. \tag{2.6}$$

The non-zero components of modified (effective) Einstein tensor for axial symmetry takes the following form

$$
\begin{aligned}
G^{00} = {} & \frac{\kappa}{A^2 f_R}\rho + \frac{1}{A^2 f_R}\left[\frac{f - R f_R}{2} - \frac{\dot{f}_R}{A^2}\left(\frac{2\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{f'_R}{B^2}\left(\frac{2B'}{B} - \frac{C'}{C} + \frac{1}{r}\right)\right. \\
& \left. - \frac{f^{\theta}_R}{r^2 B^2}\left(\frac{2B^{\theta}}{B} - \frac{C^{\theta}}{C}\right) + \frac{f''_R}{B^2}\right], \qquad (2.7)
\end{aligned}
$$

$$
G^{01} = \frac{1}{A^2 B^2 f_R}\left[\dot{f_R}' - \frac{A'}{A}\dot{f}_R - \frac{\dot{B}}{B} f'_R\right], \qquad (2.8)
$$

$$
G^{02} = \frac{1}{r^2 A^2 B^2 f_R}\left[\dot{f_R}^{\theta} - \frac{A^{\theta}}{A}\dot{f}_R - \frac{\dot{B}}{B} f^{\theta}_R\right], \qquad (2.9)
$$

$$
\begin{aligned}
G^{11} = {} & \frac{\kappa}{B^2 f_R}P_{xx} + \frac{1}{B^2 f_R}\left[-\frac{f - R f_R}{2} - \frac{\ddot{f}_R}{A^2} - \frac{f^{\theta\theta}_R}{r^2 B^2} - \frac{\dot{f}_R}{A^2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right. \\
& \left. - \frac{f'_R}{B^2}\left(\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} - \frac{1}{r}\right) - \frac{f^{\theta}_R}{r^2 B^2}\left(\frac{A^{\theta}}{A} - \frac{3B^{\theta}}{B} + \frac{C^{\theta}}{C}\right)\right], \qquad (2.10)
\end{aligned}
$$

$$
G^{12} = \frac{\kappa}{r^2 B^4 f_R}P_{xy} + \frac{1}{r^2 B^4 f_R}\left[f'^{\theta}_R - \frac{B^{\theta}}{B} f'_R - \frac{B'}{B} f^{\theta}_R\right], \qquad (2.11)
$$

$$
\begin{aligned}
G^{22} = {} & \frac{\kappa}{r^2 B^2 f_R}P_{yy} + \frac{1}{r^2 B^4 f_R}\left[-\frac{f - R f_R}{2} + \frac{\ddot{f}_R}{A^2} - \frac{\dot{f}_R}{A^2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)\right. \\
& \left. - \frac{f''_R}{B^2} - \frac{f'_R}{B^2}\left(\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C}\right) - \frac{f^{\theta}_R}{r^2 B^2}\left(\frac{A^{\theta}}{A} - \frac{B^{\theta}}{B} + \frac{C^{\theta}}{C}\right)\right], \qquad (2.12)
\end{aligned}
$$

$$
\begin{aligned}
G^{33} = {} & \frac{\kappa}{C^2 f_R}P_{zz} + \frac{1}{C^2 f_R}\left[-\frac{f - R f_R}{2} + \frac{\ddot{f}_R}{A^2} - \frac{f^{\theta\theta}_R}{r^2 B^2} - \frac{\dot{f}_R}{A^2}\left(\frac{\dot{A}}{A} - \frac{2\dot{B}}{B}\right)\right. \\
& \left. - \frac{f''_R}{B^2} - \frac{f'_R}{B^2}\left(\frac{A'}{A} - \frac{2B'}{B} - \frac{1}{r}\right) - \frac{f^{\theta}_R}{r^2 B^2}\left(\frac{A^{\theta}}{A} - \frac{2B^{\theta}}{B}\right)\right]. \qquad (2.13)
\end{aligned}
$$

Herein dot, prime and $\theta$ indicates the time, radial and axial derivatives respectively. The Ricci scalar corresponding to the metric is

$$
\begin{aligned}
R = {} & \frac{2}{A^2}\left[\frac{\dot{A}}{A}\left(\frac{2\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{\dot{B}}{B}\left(\frac{\dot{B}}{B} + \frac{2\dot{C}}{C}\right) - \frac{2\ddot{B}}{B} - \frac{\ddot{C}}{C}\right] \\
& + \frac{2}{B^2}\left[\frac{A''}{A} + \frac{A'C'}{AC} + \frac{B''}{B} - \frac{1}{r}\left(\frac{A'}{A} - \frac{B'}{B} - \frac{C'}{C}\right) - \frac{B'^2}{B^2}\right. \\
& \left. + \frac{C''}{C} + \frac{1}{r^2}\left(\frac{A^{\theta\theta}}{A} + \frac{B^{\theta\theta}}{B} + \frac{C^{\theta\theta}}{C} - \frac{B^{\theta 2}}{B^2} + \frac{A^{\theta}C^{\theta}}{AC}\right)\right]. \qquad (2.14)
\end{aligned}
$$

The dynamical equations play pivotal role in description of evolution, conservation provides basis for the collapse equation. In order to formulate collapse equation, we first

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need to develop dynamical equations by taking contracted Bianchi identities as under

$$G^{uv}_{;v}V_u=0\Rightarrow\left[\tfrac{\kappa}{f_R}T^{0v}+\tfrac{\kappa}{f_R}\overset{(D)}{T^{0v}}\right]_{;v}(-A)=0, \tag{2.15}$$

$$G^{uv}_{;v}K_u\ =0\Rightarrow\left[\tfrac{\kappa}{f_R}T^{1v}+\tfrac{\kappa}{f_R}\overset{(D)}{T^{1v}}\right]_{;v}(B)=0, \tag{2.16}$$

$$G^{uv}_{;v}L_u\ =0\Rightarrow\left[\tfrac{\kappa}{f_R}T^{2v}+\tfrac{\kappa}{f_R}\overset{(D)}{T^{2v}}\right]_{;v}(rB)=0, \tag{2.17}$$

implying

$$G^{00}_{,0}+G^{01}_{,1}+G^{02}_{,2}+G^{00}\left(\frac{2\dot{A}}{A}+\frac{2\dot{B}}{B}+\frac{\dot{C}}{C}\right)+G^{01}\left(\frac{3A'}{A}+\frac{2B'}{B}+\frac{C'}{C}+\frac{1}{r}\right)$$
$$+G^{02}\left(\frac{3A^{\theta}}{A}+\frac{2B^{\theta}}{B}+\frac{C^{\theta}}{C}\right)+G^{11}\frac{B\dot{B}}{A^2}+G^{22}\frac{r^2B\dot{B}}{A^2}+G^{33}\frac{C\dot{C}}{A^2}=0, \tag{2.18}$$

$$G^{01}_{,0}+G^{11}_{,1}+G^{12}_{,2}+G^{00}\frac{AA'}{B^2}+G^{01}\left(\frac{\dot{A}}{A}+\frac{4\dot{B}}{B}+\frac{\dot{C}}{C}\right)+G^{11}\left(\frac{A'}{A}+\frac{3B'}{B}+\frac{C'}{C}\right.$$
$$\left.+\frac{1}{r}\right)+G^{12}\left(\frac{A^{\theta}}{A}+\frac{4B^{\theta}}{B}+\frac{C^{\theta}}{C}\right)-G^{22}\left(r+\frac{r^2B'}{B}\right)+G^{33}\frac{CC'}{B^2}=0, \tag{2.19}$$

$$G^{02}_{,0}+G^{12}_{,1}+G^{22}_{,2}+G^{00}\frac{AA^{\theta}}{r^2B^2}+G^{02}\left(\frac{\dot{A}}{A}+\frac{4\dot{B}}{B}+\frac{\dot{C}}{C}\right)-\frac{B^{\theta}}{r^2B}G^{11}+\left(\frac{A'}{A}\right.$$
$$\left.+\frac{4B'}{B}+\frac{C'}{C}+\frac{3}{r}\right)G^{12}+G^{22}\left(\frac{A^{\theta}}{A}+\frac{3B^{\theta}}{B}+\frac{C^{\theta}}{C}\right)-G^{33}\frac{CC^{\theta}}{r^2B^2}=0\,. \tag{2.20}$$

The notation $0,1$ and $2$ denotes $t,r$ and $\theta$, respectively. The components of field equations given in eqs. (2.7)–(2.13) can be inserted in above three equations to view matter and effective components. In following section we present the extended Starobinsky model and also the perturbation scheme for dynamical equations.

## 3 Extended Starobinsky model and perturbation approach

We consider the supersymmetric supergravity $f(R)$ model representing $R^n$ extension of well known Starobinsky model [53]

$$f(R)=R+\alpha R^2+\beta R^n, \tag{3.1}$$

where $n\geq 3$, $\alpha$ and $\beta$ are positive quantities, considered to be positive for stable stellar configuration. The perturbation scheme has been implemented to evaluate role of the factors contributing in the establishment of instability range. Initially all the quantities are taken to be in hydrostatic equilibrium and time transition implicates the time dependence as well. First order perturbations are introduced in dynamical and field equations,



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assuming $0 < \epsilon \ll 1$

$$\begin{aligned}
A(t,r,\theta) &= A_0(r,\theta) + \epsilon D(t) a(r,\theta), && (3.2)\\
B(t,r,\theta) &= B_0(r,\theta) + \epsilon D(t) b(r,\theta), && (3.3)\\
C(t,r,\theta) &= C_0(r,\theta) + \epsilon D(t) c(r,\theta), && (3.4)\\
\rho(t,r,\theta) &= \rho_0(r,\theta) + \epsilon \bar{\rho}(t,r,\theta), && (3.5)\\
P_{xx}(t,r,\theta) &= P_{xx0}(r,\theta) + \epsilon \bar{P}_{xx}(t,r,\theta), && (3.6)\\
P_{yy}(t,r,\theta) &= P_{yy0}(r,\theta) + \epsilon \bar{P}_{yy}(t,r,\theta), && (3.7)\\
P_{zz}(t,r,\theta) &= P_{zz0}(r,\theta) + \epsilon \bar{P}_{zz}(t,r,\theta), && (3.8)\\
P_{xy}(t,r,\theta) &= P_{xy0}(r,\theta) + \epsilon \bar{P}_{xy}(t,r,\theta), && (3.9)\\
R(t,r,\theta) &= R_0(r,\theta) + \epsilon D(t) e(r,\theta), && (3.10)\\
f(R) &= \left(R_0 + \alpha R_0^2 + \beta R_0^n\right) + \epsilon D(t) e(r,\theta)\left(1\right. \\
&\quad \left. + 2\alpha R_0 + \beta n R_0^{n-1}\right), && (3.11)\\
f_R(R) &= \left(1 + 2\alpha R_0 + \beta n R_0^{n-1}\right) + \epsilon D(t) e(r,\theta)\left(2\alpha\right. \\
&\quad \left. + \beta n(n-1) R_0^{n-2}\right). && (3.12)
\end{aligned}$$

The perturbed form of dynamical equations (2.18)–(2.20) leads to the following set of equations

$$\kappa\left[\dot{\bar{\rho}} + \left\{\rho_0\left(\frac{2b}{B_0} + \frac{c}{C_0} - \frac{J}{I}\right) + \frac{b}{B_0}(P_{xx0} + P_{yy0}) + \frac{c}{C_0}P_{zz0} + \frac{Z_{1p}}{\kappa}\right\}\dot{D}\right] = 0, \qquad (3.13)$$

$$\begin{aligned}
&\kappa\left[\left(\frac{\bar{P}_{xx}}{I}\right)' + \left(\frac{\bar{P}_{xy}}{I}\right)^{\theta} + \frac{A_0'}{A_0}\bar{\rho} + D\left\{\rho_0\left(\frac{(aA_0)'}{A_0^2} + \frac{A_0'}{A_0}\right) + P_{xx0}\left(\left(\frac{a}{A_0}\right)'\right.\right.\right.\\
&\quad \left. + 3\left(\frac{b}{B_0}\right)' + \left(\frac{c}{C_0}\right)'\right) + \frac{1}{r^2}P_{xy0}\left(\left(\frac{a}{A_0}\right)^{\theta} + 4\left(\frac{b}{B_0}\right)^{\theta} + \left(\frac{c}{C_0}\right)^{\theta}\right)\\
&\quad \left. - P_{yy0}\left(\frac{b}{B_0}\right)' - P_{zz0}\left(\frac{(cC_0)'}{C_0^2} + \frac{C_0'}{C_0}\right)\right\} + \frac{\bar{P}_{xy}}{r^2}\left(\frac{A_0^{\theta}}{A_0} + \frac{3B_0^{\theta}}{B_0} + \frac{C_0^{\theta}}{C_0}\right)\\
&\quad \left. + \bar{P}_{xx}\left(\frac{A_0'}{A_0} + \frac{2B_0'}{B_0} + \frac{C_0'}{C_0} + \frac{1}{r}\right) - \bar{P}_{yy}\left(\frac{1}{r} + \frac{B_0'}{B_0}\right) - \bar{P}_{zz}\frac{C_0'}{C_0}\right]\\
&\quad + \frac{Z_{2p}}{\kappa} = 0, \qquad (3.14)
\end{aligned}$$

$$\begin{aligned}
&\kappa\left[\left(\frac{\bar{P}_{xy}}{I}\right)' + \left(\frac{\bar{P}_{yy}}{I}\right)^{\theta} + \frac{1}{r^2}\left(\frac{A_0^{\theta}}{A_0}\bar{\rho} - \frac{B_0^{\theta}}{B_0}\bar{P}_{xx} + \left(\frac{A_0^{\theta}}{A_0} + 3\frac{B_0^{\theta}}{B_0} + \frac{C_0^{\theta}}{C_0}\right)\bar{P}_{yy}\right.\right.\\
&\quad \left. + \frac{C_0^{\theta}}{C_0}\bar{P}_{zz}\right) + D\left\{\frac{1}{r^2}\left(\rho_0\left(\frac{(aA_0)^{\theta}}{A_0^2} + \frac{A_0^{\theta}}{A_0}\right) - P_{xx0}\left(\frac{b}{B_0}\right)^{\theta} - P_{zz0}\left(\frac{(cC_0)^{\theta}}{C_0^2}\right.\right.\right.\\
&\quad \left.\left. + \frac{C_0^{\theta}}{C_0}\right) + P_{yy0}\left(\left(\frac{a}{A_0}\right)^{\theta} + 3\left(\frac{b}{B_0}\right)^{\theta} + \left(\frac{c}{C_0}\right)^{\theta}\right)\right) + P_{xy0}\left(\frac{A_0'}{A_0} + \frac{4B_0'}{B_0}\right.\\
&\quad \left.\left.\left. + \frac{C_0'}{C_0} + \frac{3}{r}\right)\right\}\right] + \frac{Z_{3p}}{\kappa} = 0, \qquad (3.15)
\end{aligned}$$

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where $Z_{1p}, Z_{2p}$ and $Z_{3p}$ are perturbed dark source components of conservation equations as shown in appendix. For the sake of simplicity, we put $I = 1 + 2\alpha R_0 + \beta n R_0^{n-1}$, $J = e(2\alpha R_0 + \beta n(n-1)R_0^{n-2}$ and $L = \alpha R_0^2 + \beta(n-1)R_0^n$. Energy density $\bar{\rho}$ is derived from eq. (3.13) as

$$\dot{\bar{\rho}} = -\left\{\rho_0\left(\frac{2b}{B_0} + \frac{c}{C_0} - \frac{J}{I}\right) + \frac{b}{B_0}(P_{xx0} + P_{yy0}) + \frac{c}{C_0}P_{zz0} + \frac{1}{\kappa}Z_{1p}\right\}D. \tag{3.16}$$

The relationship between energy density and the corresponding stresses can be taken as [51, 55]

$$\bar{P}_i = \Gamma\frac{p_{i0}}{\rho_0 + p_{i0}}\bar{\rho}. \tag{3.17}$$

Here, $\Gamma$ denotes the variation of pressure stresses depending on energy density. Using eqs. (3.16) and (3.17), we have

$$\bar{P}_{xx} = -\Gamma\frac{p_{xx0}}{\rho_0 + p_{xx0}}\left\{\rho_0\left(\frac{2b}{B_0} + \frac{c}{C_0} - \frac{J}{I}\right) + \frac{b}{B_0}(P_{xx0} + P_{yy0}) + \frac{c}{C_0}P_{zz0} + \frac{1}{\kappa}Z_{1p}\right\}D, \tag{3.18}$$

$$\bar{P}_{yy} = -\Gamma\frac{p_{yy0}}{\rho_0 + p_{xx0}}\left\{\rho_0\left(\frac{2b}{B_0} + \frac{c}{C_0} - \frac{J}{I}\right) + \frac{b}{B_0}(P_{xx0} + P_{yy0}) + \frac{c}{C_0}P_{zz0} + \frac{1}{\kappa}Z_{1p}\right\}D, \tag{3.19}$$

$$\bar{P}_{zz} = -\Gamma\frac{p_{zz0}}{\rho_0 + p_{xx0}}\left\{\rho_0\left(\frac{2b}{B_0} + \frac{c}{C_0} - \frac{J}{I}\right) + \frac{b}{B_0}(P_{xx0} + P_{yy0}) + \frac{c}{C_0}P_{zz0} + \frac{1}{\kappa}Z_{1p}\right\}D, \tag{3.20}$$

$$\bar{P}_{xy} = -\Gamma\frac{p_{xy0}}{\rho_0 + p_{xx0}}\left\{\rho_0\left(\frac{2b}{B_0} + \frac{c}{C_0} - \frac{J}{I}\right) + \frac{b}{B_0}(P_{xx0} + P_{yy0}) + \frac{c}{C_0}P_{zz0} + \frac{1}{\kappa}Z_{1p}\right\}D. \tag{3.21}$$

An ordinary differential equation can be extracted from perturbed form of Ricci scalar as

$$\ddot{D}(t) - Z_4(r)D(t) = 0\,, \tag{3.22}$$

where $Z_4$ describes the perturbed quantities and is given in appendix, defined in a way that positivity holds for each term, the solution of above equation becomes

$$D(t) = -e^{\sqrt{Z_4}t}. \tag{3.23}$$

The set of dynamical equations (3.13)–(3.15) assists in formation of collapse equation. Eq. (3.13) has been used to extract expression for perturbed energy density that is further utilized to determine the pressure anisotropy. Insertion of perturbed quantities in eqs. (3.14) and (3.15) serve as the evolution equation that are identical along radial and axial coordinates and any of them can be employed to discuss stellar evolution.

## 4 Dynamical analysis of N and pN approximation

Herein, we will use eq. (3.14) as collapse equation to estimate instability range for N and pN regions by substitution of perturbed quantities found in eqs. (3.16), (3.23) and (3.18)–(3.21). In the following subsections we discuss the dynamical analysis in N and pN approximation.



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### 4.1 Newtonian limit

To arrive at N-approximation, we substitute $A_0 = 1,\ B_0 = 1,\ \rho_0 \gg p_{i0}; i = xx, yy, xy, yy$ and without loss of generality $C_0 = r$, so eq. (3.14) implies

$$\Gamma < \frac{a'\rho_0 - P_{yy0}b' - \frac{P_{zz0}}{r^2}(c'+r) + U_1 + \frac{1}{\kappa}Z_{2^N_p}}{\frac{1}{r}\left(2b + c - \frac{J}{I}\right)(2P_{xx0} + P_{yy0} + P_{zz0}) + U_2}, \tag{4.1}$$

where $Z_{2^N_p}$ are the terms of $Z_{2p}$ that belongs to N-approximation, and

$$U_1 = P_{xx0}\left(a' - 3b' - \frac{1}{r^2}\right) + P_{xy0}\left(a^\theta + 4b^\theta\right),$$
$$U_2 = \left\{\frac{P_{xx0}}{I}\left(2b + c - \frac{J}{I}\right)\right\}' + \left\{\frac{P_{xy0}}{I}(a + 4b)\right\}^\theta.$$

The inequality mentioned above express $\Gamma$ in terms of the perturbed metric and dark source terms. The gravitating system remains unstable as long as the above condition holds. Each term belonging to inequality is presumed in a way that whole expression on right side of the $\Gamma$ remains positive. Since the perturbed stresses are negative depicting the collapsing scenario, so in order to satisfy the above inequality some physical quantities are constrained as follows

$$\frac{J}{I} > 2b + c, \quad a < -4b, \quad a' < 3b' + \frac{1}{r^2}. \tag{4.2}$$

These restrictions are vital in establishment of the instability range in N-approximation.

### 4.2 Post Newtonian regime

In this approximation we take $A_0 = 1 - \frac{m_0}{r}$ and $B_0 = 1 + \frac{m_0}{r}$ implying

$$\Gamma < \frac{\frac{\rho_0}{r-m_0}U_4 + U_5 - \frac{m_0}{r(r-m_0)\kappa}Z_{1^N_p} + \frac{1}{\kappa}Z_{2^N_p}}{U_6 + \left(\frac{P_{xx0}}{I}U_5\right)' + \left(\frac{P_{xy0}}{I}U_5\right)^\theta}, \tag{4.3}$$

where

$$U_3 = \left(\frac{2br}{r+m_0} + \frac{c}{r} - \frac{J}{I}\right), \quad U_4 = a'r + \frac{m_0}{r-m_0}(a+1) - \frac{m_0}{r}U_3,$$
$$U_5 = P_{xx0}\left\{\left(\frac{ar}{r-m_0}\right)' + 3\left(\frac{br}{r+m_0}\right)' + \left(\frac{c}{r}\right)'\right\} + \frac{P_{xy0}}{r^2}\left\{\left(\frac{ar}{r-m_0}\right)^\theta\right.$$
$$\left. + 4\left(\frac{br}{r+m_0}\right)^\theta\right\} - P_{yy0}\left(\frac{br}{r+m_0}\right)' - \frac{P_{zz0}}{r}\left(\frac{c}{r} + c' + 1\right),$$
$$U_6 = \frac{U_3}{r^2}\left\{2m_0P_{xy0}\left(\frac{2m_0 - r}{r^2 - m_0^2}\right) + rP_{xx0}\left(2 + \frac{m_0(3m_0 - r)}{r^2 - m_0^2}\right) + P_{zz0}\right.$$
$$\left. + P_{yy0}\left(\frac{r - 2m_0}{r - m_0}\right)\right\}.$$

Again the quantities are restricted in order to get the instability range as under

$$\left(\frac{ar}{r-m_0}\right)' + 3\left(\frac{br}{r+m_0}\right)' + \left(\frac{c}{r}\right)' < 0, \quad \left(\frac{ar}{r-m_0}\right)^\theta + 4\left(\frac{br}{r+m_0}\right)^\theta < 0$$
$$\left(\frac{br}{r+m_0}\right)' > 0, \quad m_0 < \frac{r}{2}.$$

The self gravitating axially symmetric sources remains unstable in pN-regime until the inequality (4.2) remains valid which analytically defines the instability range. The results for some specific form of metric coefficients and the usual matter can be deduced by the induction of corresponding values in dynamical equations that further bring variations in adiabatic index accordingly.

## 5 Summary and discussion

The significance of dynamical analysis in modified gravity theories urge us to explore the dynamical instability for axial symmetry of self-gravitating systems in $f(R)$ framework. The motivation for the study of axial symmetry came from the fact that the observational gravitating systems may deviate from the most studied spherical symmetry. Obviously it is not the basic characteristic of gravitating sources, but such situation may occur incidently. Herein, we deal with the restricted class of axially symmetric sources i.e., ignoring meridional motions and motions around symmetry axis. Consequently, vorticity of sources with respect to the system vanishes for observer at rest.

The $f(R)$ model, we have taken is the extension of extensively studied Starobinsky model [52], i.e. inclusion of $n^{th}$ order term of curvature in $f(R) = R + \alpha R^2$, provided that $n \geq 3$ [53]. The extended Starobinsky model describe the supersymmetric supergravity model constructed to include more general analysis of the higher order curvature contributions. The $f(R)$ form we have chosen is viable, satisfying both viability criterion, i.e., positivity of first and second order derivatives.

The field equations for restricted axially symmetric sources with three independent metric functions are formulated in $f(R)$ gravity. The modified field equations are utilized to develop the dynamical equation for the anisotropic fluid by consideration of contacted Bianchi identities (conservation equations). The modified dynamical equations are highly complicated non-linear equations whose general solution has not been ascertained yet, that is why we have used perturbation scheme to study the dynamical system under the influence of extended Starobinsky model. Eulerian frame has been considered for the dynamical analysis, initially all the physical quantities are assumed to be in hydrostatic equilibrium. Implementation of linear perturbation on dynamical equations assists in the formation of collapse equation.

The collapse equation and substitution of the expressions for perturbed energy density, pressure anisotropy leads to the second order ordinary differential equations, which is used in estimation of adiabatic index in terms of the material and dark source components. The instability range for the N and pN regime has been established in eqs. (4.1) and (4.2), respectively. For each approximation the physical quantities are constrained in order to satisfy the stellar stable configuration. It is found that $R^n$ extension of Starobinsky model describes the more practical substitute for higher order curvature corrections. The results are analytic and so more generic, stability range for some particular scenarios can be analyzed in depth by considering numerical approach. The limiting case $\alpha \to 0, \beta \to 0$ defines the correction to GR solutions. The assumption $\beta \to 0$ corresponds to the dynamical analysis of Starobinsky model, in accordance with [51].

## A $f(R,T)$ corrections in dynamical equations

The dark Source components of the perturbed dynamical equation are given in following set of equations

$$
\begin{aligned}
Z_{1p} ={}& \frac{e\left(1-\beta n(2-n)R_0^{n-1}\right)}{2} - A_0^2\Big\{\frac{1}{A_0^2B_0^2I^2}\Big(J'\Big(1-\frac{b}{B_0}\Big)-\frac{A_0'}{A_0}J\Big)\Big\}_{,1} \\
&+\frac{LJ}{2I}-\frac{A_0^2}{r^2}\Big\{\frac{1}{A_0^2B_0^2I^2}\Big(J^\theta\Big(1-\frac{b}{B_0}\Big)-\frac{A_0^\theta}{A_0}J\Big)\Big\}_{,2}+\frac{1}{B_0^2}\Big[-\Big(\frac{2a}{A_0}+\frac{2b}{B_0}+ \\
&\frac{J}{I}\Big)\Big\{2\alpha\Big((R_0R_0')'+\frac{(R_0R_0^\theta)^\theta}{r^2}\Big)+\beta n(n-1)\Big((R_0^{n-2}R_0')'+\frac{(R_0^{n-2}R_0^\theta)^\theta}{r^2}\Big)\Big\} \\
&+I'\Big\{\Big(\frac{c}{C_0}\Big)'-2\Big(\frac{b}{B_0}\Big)'-\frac{b}{B_0}\Big(\frac{2A_0'}{A_0}+\frac{2B_0'}{B_0}-\frac{3}{r}\Big)-\frac{c}{C_0}\Big(\frac{A_0'}{A_0}-\frac{C_0'}{C_0}-\frac{1}{r}\Big) \\
&+\frac{J}{I}\Big(\frac{C_0'}{C_0}+\frac{2B_0'}{B_0}-\frac{3}{r}\Big)\Big\}+(e'J)'+\frac{(e^\theta J)^\theta}{r^2}+\frac{I^\theta}{r^2}\Big\{\Big(\frac{c}{C_0}\Big)^\theta-2\Big(\frac{b}{B_0}\Big)^\theta- \\
&\frac{b}{B_0}\Big(\frac{2A_0^\theta}{A_0}+\frac{2B_0^\theta}{B_0}\Big)-\frac{c}{C_0}\Big(\frac{A_0^\theta}{A_0}-\frac{C_0^\theta}{C_0}\Big)+\frac{J}{I}\Big(\frac{C_0^\theta}{C_0}+\frac{2B_0^\theta}{B_0}\Big)\Big\}+J'\Big(\frac{C_0'}{C_0} \\
&-\frac{2B_0'}{B_0}+\frac{1}{r}\Big)+\frac{J^\theta}{r^2}\Big(\frac{C_0^\theta}{C_0}-\frac{2B_0^\theta}{B_0}\Big)+\Big(\frac{2a}{A_0}+\frac{b}{B_0}\Big)\Big(I''+\frac{I^{\theta\theta}}{r^2}\Big)-\Big(\frac{3A_0'}{A_0} \\
&+\frac{C_0'}{C_0}+\frac{2B_0'}{B_0}+\frac{1}{r}\Big)\Big(\frac{J'}{I}\Big(1-\frac{b}{B_0}\Big)-\frac{A_0'}{A_0}\frac{J}{I}\Big)+\Big(\frac{3A_0^\theta}{A_0}+\frac{C_0^\theta}{C_0}+\frac{2B_0^\theta}{B_0}\Big)\Big(\frac{A_0^\theta}{A_0}\frac{J}{I} \\
&-\frac{J^\theta}{I}\Big(1-\frac{b}{B_0}\Big)\Big)\Big], \qquad\qquad (A.1)
\end{aligned}
$$

$$
\begin{aligned}
Z_{2p} ={}& \Big[\Big[\frac{1}{IB_0^2}\Big\{\frac{\ddot{D}}{DA_0^2}-\frac{1}{B_0^2}\Big\{\frac{NB_0^2}{2}+I'\Big(\Big(\frac{a}{A_0}\Big)'-\Big(\frac{b}{B_0}\Big)'+\Big(\frac{c}{C_0}\Big)'\Big) \\
&+\Big(J'-\frac{2b}{B_0}I'\Big)\Big(\frac{A_0'}{A_0}+\frac{C_0'}{C_0}-\frac{B_0'}{B_0}-\frac{1}{r}\Big)+\frac{1}{r^2}\Big(J^{\theta\theta}+\Big(J^\theta-\frac{2b}{B_0}I^\theta\Big)\Big(\frac{A_0^\theta}{A_0} \\
&-\frac{3B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}\Big)+\frac{2b}{B_0}I^{\theta\theta}+I^\theta\Big(\Big(\frac{a}{A_0}\Big)^\theta+\Big(\frac{c}{C_0}\Big)^\theta-3\Big(\frac{b}{B_0}\Big)^\theta\Big)\Big)\Big\}\Big\}\Big]_{,1} \\
&+\Big[\frac{1}{r^2IB_0^4}\Big\{J'^\theta+\Big(\frac{b}{B_0}\Big)^\theta I'+J^\theta\Big(\frac{B_0'}{B_0}+\frac{1}{r}\Big)-\Big(\frac{b}{B_0}\Big)'I^\theta\Big\}\Big]_{,2}\Big]IB_0^4 \\
&-N\frac{B_0'}{B_0}+\frac{A_0'}{A_0}\Big[J''+\frac{J^{\theta\theta}}{r^2}-\frac{2b}{B_0}\Big(I''+\frac{I^{\theta\theta}}{r^2}\Big)+\Big(J'-\frac{2b}{B_0}I'\Big)\Big(\frac{C_0'}{C_0}-\frac{2B_0'}{B_0} \\
&+\frac{1}{r}\Big)+I'\Big(\Big(\frac{c}{C_0}\Big)'-\Big(\frac{b}{B_0}\Big)'\Big)+\frac{1}{r^2}\Big\{I^\theta\Big(\Big(\frac{c}{C_0}\Big)^\theta-2\Big(\frac{b}{B_0}\Big)^\theta\Big)+\Big(J^\theta \\
&-\frac{2b}{B_0}I^\theta\Big)\Big(\frac{C_0^\theta}{C_0}-\frac{2B_0^\theta}{B_0}\Big)\Big\}\Big]+\Big(\frac{(aA_0)'}{A_0^2}-\frac{2b}{B_0}\frac{A_0'}{A_0}\Big)\Big(\frac{LB_0^2}{2}+I''+I'\Big(\frac{C_0'}{C_0}
\end{aligned}
$$



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$$
\begin{aligned}
&-\frac{2B_0'}{B_0}+\frac{1}{r}\bigg)+\frac{I^{\theta\theta}}{r^2}+\frac{I^\theta}{r^2}\left(\frac{C_0^\theta}{C_0}-\frac{2B_0^\theta}{B_0}\right)\bigg)-\bigg\{\frac{LB_0^2}{2}+I'\bigg(\frac{A_0'}{A_0}+\frac{C_0'}{C_0}-\frac{B_0'}{B_0}\\
&-\frac{1}{r}\bigg)+\frac{I^{\theta\theta}}{r^2}+\frac{I^\theta}{r^2}\left(\frac{A_0^\theta}{A_0}+\frac{C_0^\theta}{C_0}-\frac{3B_0^\theta}{B_0}\right)\bigg\}\left(\left(\frac{a}{A_0}\right)'+3\left(\frac{b}{B_0}\right)'+\left(\frac{c}{C_0}\right)'\right)\\
&-\left(\frac{A_0'}{A_0}+\frac{C_0'}{C_0}+\frac{3B_0'}{B_0}+\frac{1}{r}\right)\bigg\{I'\left(\left(\frac{a}{A_0}\right)'-\left(\frac{b}{B_0}\right)'+\left(\frac{c}{C_0}\right)'\right)+\bigg(\frac{A_0'}{A_0}-\frac{B_0'}{B_0}\\
&+\frac{C_0'}{C_0}-\frac{1}{r}\bigg)\left(J'-\frac{2b}{B_0}I'\right)+\frac{1}{r^2}\bigg(J^{\theta\theta}+\left(J^\theta-\frac{2b}{B_0}I^\theta\right)\left(\frac{A_0^\theta}{A_0}-\frac{3B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}\right)\\
&+\frac{2b}{B_0}I^{\theta\theta}+I^\theta\left(\left(\frac{a}{A_0}\right)^\theta+\left(\frac{c}{C_0}\right)^\theta-3\left(\frac{b}{B_0}\right)^\theta\right)\bigg)\bigg\}-\bigg[\bigg(\left(\frac{a}{A_0}\right)^\theta+\left(\frac{c}{C_0}\right)^\theta\\
&+4\left(\frac{b}{B_0}\right)^\theta\bigg)\left(I'^\theta+\frac{B_0^\theta}{B_0}I'+I^\theta\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)\right)-\left(\frac{A_0^\theta}{A_0}+\frac{4B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}\right)\bigg(\frac{B_0^\theta}{B_0}J'\\
&-J'^\theta-I'\left(\frac{b}{B_0}\right)^\theta-J^\theta\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)+I^\theta\left(\frac{b}{B_0}\right)'\bigg)\bigg]\frac{1}{r^2}-\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)\bigg[\frac{B_0^2}{A_0^2}\frac{\ddot D}{D}J\\
&-J''+\frac{2b}{B_0}I''-I'\left(\left(\frac{a}{A_0}\right)'-\left(\frac{b}{B_0}\right)'+\left(\frac{c}{C_0}\right)'\right)-\left(J'-\frac{2b}{B_0}I'\right)\bigg(\frac{A_0'}{A_0}+\frac{C_0'}{C_0}\\
&-\frac{B_0'}{B_0}\bigg)+\frac{1}{r^2}\bigg(I^\theta\left(\left(\frac{a}{A_0}\right)^\theta+\left(\frac{c}{C_0}\right)^\theta-\left(\frac{b}{B_0}\right)^\theta\right)-\left(J^\theta-\frac{2b}{B_0}I^\theta\right)\bigg(\frac{A_0^\theta}{A_0}-\frac{B_0^\theta}{B_0}\\
&+\frac{C_0^\theta}{C_0}\bigg)\bigg)\bigg]+\left(\frac{b}{B_0}\right)'\bigg[\frac{LB_0^2}{2}+I'\left(\frac{A_0'}{A_0}+\frac{C_0'}{C_0}-\frac{B_0'}{B_0}\right)-\frac{I^\theta}{r^2}\left(\frac{A_0^\theta}{A_0}-\frac{B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}\right)\\
&-I''\bigg]+\frac{C_0'}{C_0}\bigg[J''-\frac{2b}{B_0}I''+I'\left(\left(\frac{a}{A_0}\right)'-\left(\frac{2b}{B_0}\right)'\right)+\left(J'-\frac{2b}{B_0}I'\right)\bigg(\frac{A_0'}{A_0}-\frac{B_0'}{B_0}\\
&+\frac{1}{r}\bigg)+\frac{1}{r^2}\bigg\{J^{\theta\theta}+\left(J^\theta-\frac{2b}{B_0}I^\theta\right)\left(\frac{A_0^\theta}{A_0}-\frac{2B_0^\theta}{B_0}\right)+I^\theta\left(\left(\frac{a}{A_0}\right)^\theta-2\left(\frac{b}{B_0}\right)^\theta\right)\\
&-\frac{2b}{B_0}I^{\theta\theta}\bigg\}\bigg]+\left(\frac{(cC_0)'}{C_0^2}-\frac{2b}{B_0}\frac{C_0'}{C_0}\right)\bigg[I'\left(\frac{A_0'}{A_0}-\frac{2B_0'}{B_0}+\frac{1}{r}\right)-\frac{I^\theta}{r^2}\left(\frac{A_0^\theta}{A_0}-\frac{2B_0^\theta}{B_0}\right)\\
&+\frac{LB_0^2}{2}+I''+\frac{I^{\theta\theta}}{r^2}\bigg]-\frac{\ddot D B_0^2}{DA_0^2I}\left(J'-\frac{A_0'}{A_0}J-\frac{b}{B_0}I'\right), \qquad\qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
Z_{3p}=Ir^2B_0^4&\bigg[\bigg[\frac{1}{r^2IB_0^4}\bigg\{J'^\theta+\left(\frac{b}{B_0}\right)^\theta I'+J^\theta\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)-\left(\frac{b}{B_0}\right)'I^\theta\bigg\}\bigg]_{,1}\\
&+\frac{\ddot DB_0^2}{DA_0^2I}\left(\frac{A_0^\theta}{A_0}J+\frac{b}{B_0}I^\theta-J^\theta\right)+\bigg[\frac{1}{Ir^2B_0^4}\bigg\{\frac{\ddot DB_0^2}{DA_0^2}J-J''+\frac{2b}{B_0}I''+\bigg(\frac{2b}{B_0}I'\\
&-J'\bigg)\left(\frac{A_0'}{A_0}+\frac{C_0'}{C_0}-\frac{B_0'}{B_0}\right)-I'\left(\left(\frac{a}{A_0}\right)'-\left(\frac{b}{B_0}\right)'+\left(\frac{c}{C_0}\right)'\right)+\frac{1}{r^2}\bigg(\bigg(\frac{2b}{B_0}I^\theta\\
&-J^\theta\bigg)\left(\frac{A_0^\theta}{A_0}-\frac{B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}\right)-I^\theta\left(\left(\frac{a}{A_0}\right)^\theta+\left(\frac{c}{C_0}\right)^\theta-\left(\frac{b}{B_0}\right)^\theta\right)\bigg)\bigg\}\bigg]_{,2}\bigg]
\end{aligned}
$$

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$$
\begin{aligned}
&- N\frac{B_0^\theta}{B_0} + \frac{A_0^\theta}{A_0}\Bigg[J'' + \frac{J^{\theta\theta}}{r^2} - \frac{2b}{B_0}\left(I'' + \frac{I^{\theta\theta}}{r^2}\right) + \left(J' - \frac{2b}{B_0}I'\right)\Bigg(\frac{C_0'}{C_0} - \frac{2B_0'}{B_0} \\
&+ \frac{1}{r}\Bigg) + I'\left(\left(\frac{c}{C_0}\right)' - \left(\frac{b}{B_0}\right)'\right) + \frac{1}{r^2}\Bigg\{I^\theta\left(\left(\frac{c}{C_0}\right)^\theta - 2\left(\frac{b}{B_0}\right)^\theta\right) + \Bigg(J^\theta \\
&- \frac{2b}{B_0}I^\theta\Bigg)\left(\frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0}\right)\Bigg\}\Bigg] + \left(\frac{(aA_0)^\theta}{A_0^2} - \frac{2b}{B_0}\frac{A_0^\theta}{A_0}\right)\Bigg(\frac{LB_0^2}{2} + I'' + I'\Bigg(\frac{C_0'}{C_0} \\
&- \frac{2B_0'}{B_0} + \frac{1}{r}\Bigg) + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2}\left(\frac{C_0^\theta}{C_0} - \frac{2B_0^\theta}{B_0}\right)\Bigg) - \left(\frac{b}{B_0}\right)^\theta\Bigg\{\frac{LB_0^2}{2} + I'\Bigg(\frac{A_0'}{A_0} + \frac{C_0'}{C_0} \\
&- \frac{B_0'}{B_0} + \frac{1}{r}\Bigg) + \frac{I^{\theta\theta}}{r^2} + \frac{I^\theta}{r^2}\left(\frac{A_0^\theta}{A_0} + \frac{C_0^\theta}{C_0} - \frac{3B_0^\theta}{B_0}\right)\Bigg\} - \frac{B_0^\theta}{B_0}\Bigg\{I'\Bigg(\left(\frac{a}{A_0}\right)' - \left(\frac{b}{B_0}\right)' \\
&+ \left(\frac{c}{C_0}\right)'\Bigg) + \left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{C_0'}{C_0} - \frac{1}{r}\right)\left(J' - \frac{2b}{B_0}I'\right) + \frac{1}{r^2}\Bigg(J^{\theta\theta} - \Bigg(\frac{2b}{B_0}I^\theta - \\
&J^\theta\Bigg)\left(\frac{A_0^\theta}{A_0} - \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0}\right) + \frac{2b}{B_0}I^{\theta\theta} + I^\theta\left(\left(\frac{a}{A_0}\right)^\theta + \left(\frac{c}{C_0}\right)^\theta - 3\left(\frac{b}{B_0}\right)^\theta\right)\Bigg)\Bigg\} \\
&- \frac{1}{r^2}\Bigg[\left(\left(\frac{a}{A_0}\right)' + \left(\frac{c}{C_0}\right)' + \left(\frac{b}{B_0}\right)'\right)\left(I'^\theta + \frac{B_0^\theta}{B_0}I' + I^\theta\left(\frac{B_0'}{B_0} + \frac{1}{r}\right)\right) - \Bigg(\frac{A_0'}{A_0} \\
&+ \frac{4B_0'}{B_0} + \frac{C_0'}{C_0}\Bigg)\left(\frac{B_0^\theta}{B_0}J' - J'^\theta - I'\left(\frac{b}{B_0}\right)^\theta - J^\theta\left(\frac{B_0'}{B_0} + \frac{1}{r}\right) + I^\theta\left(\frac{b}{B_0}\right)'\right)\Bigg] \\
&+ \left(\frac{A_0^\theta}{A_0} + \frac{3B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0}\right)\Bigg[\frac{B_0^2}{A_0^2}\frac{\ddot{D}}{D}J + \frac{2b}{B_0}I'' - I'\left(\left(\frac{a}{A_0}\right)' - \left(\frac{b}{B_0}\right)' + \left(\frac{c}{C_0}\right)'\right) \\
&- J'' - \left(J' - \frac{2b}{B_0}I'\right)\left(\frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0}\right) - \frac{1}{r^2}\Bigg(\left(J^\theta - \frac{2b}{B_0}I^\theta\right)\Bigg(\frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} \\
&+ \frac{C_0^\theta}{C_0}\Bigg) - I^\theta\left(\left(\frac{a}{A_0}\right)^\theta + \left(\frac{c}{C_0}\right)^\theta - \left(\frac{b}{B_0}\right)^\theta\right)\Bigg)\Bigg] + \Bigg(\left(\frac{a}{A_0}\right)^\theta + \left(\frac{c}{C_0}\right)^\theta \\
&+ 3\left(\frac{b}{B_0}\right)^\theta\Bigg)\left[\frac{LB_0^2}{2} + I'\left(\frac{A_0'}{A_0} + \frac{C_0'}{C_0} - \frac{B_0'}{B_0}\right) - \frac{I^\theta}{r^2}\left(\frac{A_0^\theta}{A_0} - \frac{B_0^\theta}{B_0} + \frac{C_0^\theta}{C_0}\right) - I''\right] \\
&- \frac{C_0^\theta}{C_0}\Bigg[J'' - \frac{2b}{B_0}I'' + I'\left(\left(\frac{a}{A_0}\right)' - \left(\frac{2b}{B_0}\right)'\right) + \left(J' - \frac{2b}{B_0}I'\right)\Bigg(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} \\
&+ \frac{1}{r}\Bigg) + \frac{1}{r^2}\Bigg\{\left(J^\theta - \frac{2b}{B_0}I^\theta\right)\left(\frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0}\right) + I^\theta\left(\left(\frac{a}{A_0}\right)^\theta - 2\left(\frac{b}{B_0}\right)^\theta\right) \\
&+ J^{\theta\theta} - \frac{2b}{B_0}I^{\theta\theta}\Bigg\}\Bigg] + \left(\frac{(cC_0)^\theta}{C_0^2} - \frac{2b}{B_0}\frac{C_0^\theta}{C_0}\right)\Bigg[\frac{LB_0^2}{2} + I'\left(\frac{A_0'}{A_0} - \frac{2B_0'}{B_0} + \frac{1}{r}\right) \\
&+ I'' + \frac{I^{\theta\theta}}{r^2} - \frac{I^\theta}{r^2}r\left(\frac{A_0^\theta}{A_0} - \frac{2B_0^\theta}{B_0}\right)\Bigg],
\end{aligned}
\tag{A.3}
$$

where, $N = e(1-\beta n(n-1))R_0^{n-1}$

$$Z_4 = \frac{A_0^2}{2}\left(\frac{B_0C_0}{bC_0-cB_0}\right)\left[\frac{2}{B_0^2}\left\{\frac{A_0'C_0'}{A_0C_0}\left(\frac{a'}{A_0'}-\frac{a}{A_0}+\frac{c'}{C_0'}-\frac{c}{C_0}\right)+\frac{A_0''}{A_0}\left(\frac{a''}{A_0''}\right.\right.\right.$$
$$\left.-\frac{a}{A_0}\right)+\frac{B_0''}{B_0}\left(\frac{b''}{B_0''}-\frac{b}{B_0}\right)+\frac{C_0''}{C_0}\left(\frac{c''}{C_0''}-\frac{c}{C_0}\right)-\frac{1}{r}\left(\frac{a}{A_0}-\frac{b}{B_0}-\frac{c}{C_0}\right)'$$
$$-\frac{2B_0'}{B_0}\left(\frac{b}{B_0}\right)'+\frac{2}{r^2}\left\{\frac{2B_0^\theta}{B_0}\left(\frac{b}{B_0}\right)^\theta+\frac{A_0^{\theta\theta}}{A_0}\left(\frac{a^{\theta\theta}}{A_0^{\theta\theta}}-\frac{a}{A_0}\right)+\frac{B_0^{\theta\theta}}{B_0}\left(\frac{b^{\theta\theta}}{B_0^{\theta\theta}}-\frac{b}{B_0}\right)\right.$$
$$\left.\left.\left.+\frac{C_0^{\theta\theta}}{C_0}\left(\frac{c^{\theta\theta}}{C_0^{\theta\theta}}-\frac{c}{C_0}\right)+\frac{A_0^\theta C_0^\theta}{A_0C_0}\left(\frac{a^\theta}{A_0^\theta}-\frac{a}{A_0}+\frac{c^\theta}{C_0^\theta}-\frac{c}{C_0}\right)\right\}\right\}-e-\frac{2bR_0}{B_0}\right]. \tag{A.4}$$

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