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Hybrid Numerical Method for Heat Equation with Nonlocal Boundary Conditions in Parallel Computing Environment

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Abstract: A numerical method is developed for solving parabolic partial differential equations with integral boundary conditions. The method is moderately sixth-order accurate due to merging of sixth order finite difference scheme and fifth order Pade's approximation. Simpson's 1/3 rule is used to approximate integral conditions. The method does not involve the use of complex arithmetic and optimizes the results. It is observed that this numerical method can be easily coded on serial as well as parallel computers.

Keywords: Integral boundary conditions, method of lines, Pade's approximations, parallel algorithm, Simpson's 1/3 rule

INTRODUCTION

The need of both the scientific and the business communities' for ever growing computing supremacy led to vivid up grading in computer structural design. Most of the attempts concentrated on attaining high performance on a single processor, but recently it has been observed that attempts are being made to wrap multiple processors. Multiprocessor systems consist of a number of connected processors each of which is capable of performing compound tasks autonomously. In a sequential algorithm all tasks are carried out by a single processor but in a parallel algorithm autonomous components of the program are performed by varied processors simultaneously which save a lot of time.

In many developing countries, scientists and engineers are facing problems, when high computations and/or large memory storage is required. This is due to the lack of advance computing resources. In order to resolve such problems, the numerical method is proposed. Partial differential equations arise in many real life problems like thermo elasticity (Day, 1982), dynamics of ground water (Nakhushev, 1982) and pseudo-parabolic water transfer (Vodakhova, 1982). In the family of partial differential equations, one of the most important classes is parabolic partial differential equations with nonlocal boundary conditions. This class was studied by different authors (Wang and Liu, 1989; 1990; Muravei and Philinovskii, 1982; Liu, 1999; Aug, 2002; Deghan, 2003, 2005; Rehman and Taj, 2009) in different ways to solve such model problems numerically. This study aims at exploring one dimensional non-homogeneous heat equation with integral boundary conditions. The idea of mixed order numerical method presented here was firstly introduced by Rehman *et al*. (2012) and now it is proposed to be the best candidate for numerical solution of nonlocal problems.

Actual concept behind the use of finite-difference methods for obtaining the approximate solution of a given PDE is to approximate the derivatives appearing in the equation by a set of values of the function at a selected number of points.

Consider one dimensional heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x,t)\,, 0 < x < X, t > 0 \tag{1}$$

Subject to the given initial condition:

$$u(x,0) = F(x), 0 \leq \mathrm{x} \leq 1 \tag{2}$$

and the non-local boundary conditions:

$$u(0,t) = \int_0^1 \varphi(x,t)u(x,t)dx + G_1(t), \quad T \geq t > 0 \tag{3}$$

$$u(1,t) = \int_0^1 \tau(x,t)u(x,t)dx + G_2(t), \quad T \geq t > 0 \tag{4}$$

where, $F, G_1, G_2, \tau, \varphi$ and Q are known functions and are assumed to be sufficiently smooth to produce a smooth solution of u. T is given positive constant.

DISCRETIZATION AND HANDLING OF NONLOCAL CONDITIONS

Selecting a positive odd integer $N \geq 9$ and dividing the interval $[0, 1]$ into $N + 1$ subintervals each

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of width h, so that $(N+1)\,h = X$ and the time variable t into time steps each of length l gives a rectangular mesh of points with co-ordinates $(x_m, t_n) = (mh, nl)$ $(m = 0, 1, 2, \dots, N, N+1)$ and $(n = 0, 1, 2, \dots)$ covering the region $R = [0 < x < X] \times [t > 0]$ and its boundary ∂R consisting of the lines $x = 0, x = 1$ and $t = 0$.

Let $u\,(x,t)$ is minimum nine times continuously differentiable with respect to space variable x and further these derivatives are uniformly bounded, the space derivative in (1) may be approximated to the sixth-order accuracy at some general point (x,t) of the mesh by the expression:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{180\,h^2}\{2\mathrm{u}(\mathrm{x}-3\mathrm{h},\mathrm{t}) - 27\mathrm{u}(\mathrm{x}-2\mathrm{h},\mathrm{t}) + 270\mathrm{u}(\mathrm{x}-\mathrm{h},\mathrm{t}) - 490\mathrm{u}(\mathrm{x},\mathrm{t}) + 270\mathrm{u}(\mathrm{x}+\mathrm{h},\mathrm{t}) - 27\mathrm{u}(\mathrm{x}+2\mathrm{h},\mathrm{t}) + 2\,\mathrm{u}(\mathrm{x}+3\mathrm{h},\mathrm{t})\} - \frac{h^6}{560}\frac{\partial^8 u(x,t)}{\partial x^8} + \mathrm{O}(\mathrm{h}^7) \quad (5)$$

However, Eq. (5) is valid only for $(\mathrm{x},\mathrm{t}) = (x_m, t_n)$ with $m = 3, 4, \dots, N-2$ and $= 0, 1, 2, 3, \dots$. To get the accuracy at the same points at (x_1, t_n), (x_2, t_n), (x_{n-1}, t_n) and (x_n, t_n) special formulas must be developed which approximate $\partial^2 u\,(x,t)/\partial x^2$ not only to sixth-order but also with dominant error term $-(h^6/560)(\partial^8 u(x,t)/\partial x^8)$. It can be clearly shown that the desired approximations to $\partial^2 u(x,t)/\partial x^2$ are:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{180\,h^2}\{117\mathrm{u}(\mathrm{x}-\mathrm{h},\mathrm{t}) + 2\mathrm{u}(\mathrm{x},\mathrm{t}) - 738\mathrm{u}(\mathrm{x}+\mathrm{h},\mathrm{t}) + 1359\mathrm{u}(\mathrm{x}+2\mathrm{h},\mathrm{t}) - 1300\,\mathrm{u}(\mathrm{x}+3\mathrm{h},\mathrm{t}) + 828\,\mathrm{u}(\mathrm{x}+4\mathrm{h},\mathrm{t}) - 342\,\mathrm{u}(\mathrm{x}+5\mathrm{h},\mathrm{t}) + 83\,\mathrm{u}(\mathrm{x}+6\mathrm{h},\mathrm{t}) - 9\mathrm{u}(\mathrm{x}+7\mathrm{h},\mathrm{t})\} - \frac{h^6}{560}\frac{\partial^8 u(x,t)}{\partial x^8} + \mathrm{O}(\mathrm{h}^7) \quad (6)$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{180\,h^2}\{-9\mathrm{u}(\mathrm{x}-2\mathrm{h},\mathrm{t}) + 198\mathrm{u}(\mathrm{x}- \mathrm{h},\mathrm{t}-322\ \mathrm{u}(\mathrm{x},\mathrm{t}) + 18u(x+h,t) + 225u(x+2h,t) - 166\mathrm{u}(\mathrm{x}+3\mathrm{h},\mathrm{t}) + 72\mathrm{u}(\mathrm{x}+4\mathrm{h},\mathrm{t}) - 18\mathrm{u}(\mathrm{x}+ 5\mathrm{h},\mathrm{t} + 2\mathrm{u}(\mathrm{x}+6\mathrm{h},\mathrm{t})\} - \frac{h^6}{560}\frac{\partial^8 u(x,t)}{\partial x^8} + \mathrm{O}(\mathrm{h}^7) \quad (7)$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{180\,h^2}\{-9\mathrm{u}(\mathrm{x}+2\mathrm{h},\mathrm{t}) + 198\mathrm{u}(\mathrm{x}+ \mathrm{h},\mathrm{t}-322\ \mathrm{ux},\mathrm{t}+18\ \mathrm{ux}-\mathrm{h},\mathrm{t}+225\mathrm{ux}-2\mathrm{h},\mathrm{t}-166\ \mathrm{u}(\mathrm{x}-3\mathrm{h},\mathrm{t}) + 72\,\mathrm{u}(\mathrm{x}-4\mathrm{h},\mathrm{t}) - 18\,\mathrm{u}(\mathrm{x}-5\mathrm{h},\mathrm{t}) + 2\mathrm{u}(\mathrm{x}-6\mathrm{h},\mathrm{t})\} - \frac{h^6}{560}\frac{\partial^8 u(x,t)}{\partial x^8} + \mathrm{O}(\mathrm{h}^7) \quad (8)$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{180\,h^2}\{117\mathrm{u}(\mathrm{x}+\mathrm{h},\mathrm{t}) + 2\,\mathrm{u}(\mathrm{x},\mathrm{t}) - 738\,\mathrm{u}(\mathrm{x}-\mathrm{h},\mathrm{t}) + 1359\mathrm{u}(\mathrm{x}-2\mathrm{h},\mathrm{t}) - 1300\,\mathrm{u}(\mathrm{x}-3\mathrm{h},\mathrm{t}) + 828\,\mathrm{u}(\mathrm{x}-4\mathrm{h},\mathrm{t}) - 342\,\mathrm{u}(\mathrm{x}-5\mathrm{h},\mathrm{t}) + 83\mathrm{u}(\mathrm{x}-6\mathrm{h},\mathrm{t}) - 9\mathrm{u}(\mathrm{x}-7\mathrm{h},\mathrm{t})\} - \frac{h^6}{560}\frac{\partial^8 u(x,t)}{\partial x^8} + \mathrm{O}(\mathrm{h}^7) \quad (9)$$

at the mesh points $(x_1, t_n), (x_{n-2}, t_n), (x_{n-1}, t_n)$ and (x_n, t_n) respectively (Rehman *et al*., 2012). Appling (1) with (5)-(9) to the mesh points of the grid at time level $t = t_n$ produces asystem of ordinary differential equations of N equations and $N+2$ unknowns $U_0, U_1, U_2, \dots, U_{N+1}$. The integral term in (3) and (4) are approximated by using Simpson's $1/3$ rule as:

$$u(0,t) = \frac{h}{3}\{\varphi(0,t)u(0,t) + 4\sum_{i=1}^{\frac{N+1}{2}} \varphi((2i- 1h, tu2i-1h, t + 2\sum_{i=1}^{\frac{N+1}{2}-1} \varphi((2i)h,t)u((2i)h,t) + \varphi((N+1)h,t)u((N+1)h,t)\} + G_1(t) \quad (10)$$

$$u(1,t) = \frac{h}{3}\{\tau(0,t)u(0,t) + 4\sum_{i=1}^{\frac{N+1}{2}} \tau((2i- 1h, tu2i-1h, t + 2\sum_{i=1}^{\frac{N+1}{2}-1} \tau((2i)h,t)u((2i)h,t) + \tau((N+ 1h, tuN+1h, t\}+G2(t) \quad (11)$$

Solving Eq. (10) and (11) for U_0 and U_{N+1} and substituting these values in the above system of N linear ordinary differential equations which can be written in vector matrix form as:

$$\frac{dU(t)}{dt} = AU(t) + v(t), t > 0 \quad (12)$$

With initial distribution:

$$U(0) = F \quad (13)$$

In which $U(t) = [U_1(t), U_2(t), \dots, U_N(t)]^T$, $F = [F(x_1),\ F(x_2), \dots, F(x_N)]^T$, T denotes transpose and matrix A of order $n \times n$ is given by:

$$A = \frac{1}{180\,h^2}\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \cdots & & \lambda_{N-1} & \lambda_N \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \cdots & & \eta_{N-1} & \eta_N \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \cdots & & \varepsilon_{N-1} & \varepsilon_N \\ 2 & -27 & 270 & -490 & 270 & -27 & 2 & & \\ & 2 & -27 & 270 & -490 & 270 & -27 & 2 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 2 & -27 & 270 & -490 & 270 & -27 & 2 \\ \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 & \cdots & & \kappa_{N-1} & \kappa_N \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots & & \mu_{N-1} & \mu_N \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \cdots & & \xi_{N-1} & \xi_N \end{bmatrix}$$

where,

$\lambda_1 = 117\,m_1 + 2$, $\lambda_2 = 117\,m_2 - 738$, $\lambda_3 = 117\,m_3 + 1359$, $\lambda_4 = 117\,m_4 - 1300$,
$\lambda_5 = 117\,m_5 + 828$, $\lambda_6 = 117\,m_6 - 342$,
$\lambda_7 = 117\,m_7 + 83$, $\lambda_8 = 117\,m_8 - 9$ and $\lambda_i \geq 117 m_i\ for\ i \geq 9$
$\eta_1 = -9\,m_1 + 198$, $\lambda_2 = -9\,m_2 - 322$, $\eta_3 = -9\,m_3 + 18$, $\eta_4 = -9\,m_4 + 225$, $\eta_5 = -9\,m_5 - 166$,
$\eta_6 = -9\,m_6 - 18$, $\eta_7 = -9\,m_7 - 18$, $\eta_8 = -9\,m_8 + 2$ and $\eta_i \geq -9 m_i\ for\ i \geq 9$
$\varepsilon_1 = 2\,m_1 - 27$, $\varepsilon_2 = 2\,m_2 + 270$, $\varepsilon_3 = 2\,m_3 - 490$, $\varepsilon_4 = 2\,m_4 + 270$, $\varepsilon_5 = 2\,m_5 - 27$,
$\varepsilon_6 = 2\,m_6 + 2$ and $\varepsilon_i \geq 2 m_i\ for\ i \geq 7$
$\kappa_{N-5} = 2\,n_{N-5} + 2$, $\kappa_{N-4} = 2\,n_{N-4} - 27$,
$\kappa_{N-3} = 2\,n_{N-3} + 270$, $\kappa_{N-2} = 2\,n_{N-2} - 490$,
$\kappa_{N-1} = 2\,n_{N-1} - 270$,
$\kappa_N = 2\,n_N - 27$ and $\kappa_i \geq 2 n_i\ for 1 \leq i \leq N-6$
$\mu_{N-7} = -9\,n_{N-7} + 2$, $\mu_{N-6} = -9\,n_{N-6} - 18$,
$\mu_{N-5} = -9\,n_{N-5} + 72$, $\mu_{N-4} = -9\,n_{N-4} - 166$,
$\mu_{N-3} = -9\,n_{N-3} + 225$, $\mu_{N-2} = -9\,n_{N-2} + 18$,
$\mu_{N-1} = -9\,n_{N-1} - 322$, $\mu_N = -9\,n_N - 270$
and $\mu_i \geq -9 n_i\ for 1 \leq i \leq N-8$
$\xi_{N-7} = 117\,n_{N-7} - 9$, $\xi_{N-6} = 117\,n_{N-6} + 83$,
$\xi_{N-5} = 117\,n_{N-5} - 342$, $\xi_{N-4} = 117\,n_{N-4} + 828$,
$\xi_{N-3} = 117\,n_{N-3} - 1300$, $\xi_{N-2} = 117\,n_{N-2} + 1359$, $\xi_{N-1} = 117\,n_{N-1} - 738$, $\xi_N = 117\,n_N + 2$
and $\xi_i \geq 117 n_i\ for 1 \leq i \leq N-8$

in which:

$$m_i = \begin{cases} \dfrac{4\dfrac{h}{3}(c_4\varphi_i - c_2\tau_i)}{c_1c_4 - c_2c_3}, fori = 1, 2, 3, \ldots, N \\ \dfrac{2\dfrac{h}{3}(c_4\varphi_i - c_2\tau_i)}{c_1c_4 - c_2c_3}, fori = 2, 3, \ldots, N-1 \end{cases}$$

and

$$n_i = \begin{cases} \dfrac{4\dfrac{h}{3}(c_3\varphi_i - c_1\tau_i)}{c_2c_3 - c_1c_4}, fori = 1, 2, 3, \ldots, N \\ \dfrac{2\dfrac{h}{3}(c_3\varphi_i - c_1\tau_i)}{c_2c_3 - c_1c_4}, fori = 1, 2, 3, \ldots, N-1 \end{cases}$$

Here $c_1 = 1 - \frac{h}{3}\varphi_0$, $c_2 = -\frac{h}{3}\varphi_{N+1}$, $c_3 = -\frac{h}{3}\tau_0$, $c_4 = 1 - \frac{h}{3}\tau_{N+1}$ also $\varphi_i = \varphi(ih, t)$ and $\tau_i = \tau(ih, t)$.

The column vector $v(t)$ contains the contribution of the functions Q, G_1 and G_2 and is given as:

$$v(t) = [\tfrac{117\,l_1}{180\,h^2} + Q_1, \tfrac{-9\,l_1}{180\,h^2} + Q_2, \tfrac{2\,l_1}{180\,h^2} + Q_3, Q_4, \ldots, Q_{N-3}, \tfrac{2\,l_2}{180\,h^2} + Q_{N-2}, \tfrac{-9\,l_2}{180\,h^2} + Q_{N-1}, \tfrac{117\,l_2}{180\,h^2} + Q_N] \quad (14)$$

where,

$$l_1 = \frac{c_4G_1(t) - c_2G_2(t)}{c_1c_4 - c_2c_3} \text{ and } l_2 = \frac{c_1G_2(t) - c_3G_1(t)}{c_1c_4 - c_2c_3}$$

The solution of the system (12) subject to (13) is given by:

$$U(t) = \exp(lA)\, f + \int_0^t \exp\big((t-s)A\big)v(s)\, ds;\ t \geq 0 \quad (15)$$

which satisfies the recurrence relation:

$$U(t+l) = \exp(lA)\, U(t) + \int_t^{t+l} \exp\big((t + l - sAvsds; t=0,l,2l \quad (16)$$

To approximate the matrix exponential function in (16), fifth-order Pade's approximation, for a real scalar θ, given by:

$$E_5(\theta) = \frac{1 + b_1\theta + b_2\theta^2 + b_3\theta^3 + b_4\theta^4}{1 - a_1\theta + a_2\theta^2 - a_3\theta^3 + a_4\theta^4 - a_5\theta^5} \quad (17)$$

where,

$$a_5 = \sum_{k=1}^{4}(-1)^k \frac{a_k}{(5-k)!} \quad (18)$$

and

$$b_k = \sum_{i=1}^{k}(-1)^i \frac{a_i}{(k-i)!}, k = 0, 1, 2, 3, 4 \quad (19)$$

For stability of method, a_i $(i = 1, 2, 3, 4)$ should satisfy the following conditions:

$$a_1 > \frac{1}{2!}$$

$$a_2 > \frac{a_1}{2!} - \frac{1}{3!}$$

$$a_3 > \frac{a_2}{2!} - \frac{a_1}{3!} + \frac{1}{4!}$$

$$a_4 > \frac{a_3}{2!} - \frac{a_2}{3!} - \frac{a_1}{4!} + \frac{1}{5!}$$

The integral term in (16) is approximated as:

$$\int_t^{t+l} \exp\big((t + l - s)A\big)v(s)\, ds = W_1v(s_1) + W_2v(s_2) + W_3v(s_3) + W_4v(s_4) + W_5v\,(s_5) \quad (20)$$

where, $s_1 \neq s_2 \neq s_3 \neq s_4 \neq s_5$ and $W_i (i = 1, 2, 3, 4, 5)$ are matrices. We have:

$$\int_t^{t+l} \exp\big((t + l - s)A\big) s^{k-1}\, ds = \sum_{j=1}^{s} s_j^{k-1} W_j = M_k,\ k = 1, 2, 3, 4, 5 \tag{21}$$

with

$$M_k = A^{-1}\{t^{k-1} \exp(lA) - (t + l)^{k-1} I + (k-1) M_{k-1},\ k = 1, 2, 3, 4, 5 \tag{22}$$

Taking $s_1 = t, s_2 = t + \frac{l}{4}, \quad s_3 = t + \frac{l}{2} +, \quad s_4 = t + \frac{3\,l}{4}, s_5 = t + l$. Using $\theta = lA$ in (17) and taking $\exp(lA) = PR^{-1}$, we have (11):

$$P = (I - a_1 lA + a_2 l^2 A^2 - a_3 l^3 A^3 + a_4 l^4 A^4 - a_5 l^5 A^5)^{-1} \tag{23}$$

And

$$R = I + b_1 lA + b_2 l^2 A^2 + a_3 l^3 A^3 + a_4 l^4 A^4 \tag{24}$$

$$W_1 = \frac{l}{360}\{28I + (668 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4) lA + (-21 + 100a_1 - 260a_2 + 1920a_4) l^2 A^2 + (18 - 75a_1 + 240a_2 - 540a_3 + 720a_4) l^3 A^3\} P \tag{25}$$

$$W_2 = \frac{2l}{45}\{8I + (-154 + 760a_1 - 2880a_2 + 7680a_3) lA + (1 + 10a_1 - 100a_2 + 480a_3 - 1200a_4) l^2 A^2 + (-1 + 5a_1 - 20a_2 + 60a_3 - 120a_4) l^3 A^3\} P \tag{26}$$

$$W_3 = \frac{l}{30}\{4I + (322 - 1540a_1 + 570a_2 - 15360a_3 + 23040a_4) lA + (23 - 130a_1 + 580a_2 - 1920a_3 + 3840a_4) l^2 A^2 + (3 - 15a_1 + 60a_2 - 180a_3 + 360a_4) l^3 A^3\} P \tag{27}$$

$$W_4 = \frac{2l}{45}\{8I + (-158 + 760a_1 - 2880a_2 + 7680a_3 - 11520a_4) lA + (-21 + 110a_1 - 460a_2 + 1440a_3 - 2640a_4) l^2 A^2 + (-3 + 15a_1 - 60a_2 + 180a_3 - 360a_4) l^3 A^3\} \mathrm{P} \tag{28}$$

$$W_5 = \frac{l}{360}\{28I + (640 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4) lA + (125 - 640a_1 + 2620a_2 - 7680a_3 + 13440a_4) l^2 A^2 + (25 - 125a_1 + 500a_2 - 1500a_3 + 2640a_4) l^3 A^3 + (3 - 15a_1 + 60a_2 - 180a_3 + 360a_4) l^4 A^4\} \mathrm{P} \tag{29}$$

ALGORITHM

Assuming that $r_1, r_2, r_3, r_4, r_5\ (r_i \neq 0)$ are real distinct zeros of $R(\theta)$, the denominator of $E_5(\theta)$, then:

$$P^{-1} = \prod_{i=1}^{5} (I - \frac{l}{r_i} A) \tag{30}$$

$$\exp(lA) = p_j \sum_{j=1}^{5} (I - \frac{l}{r_i} A)^{-1} \tag{31}$$

where,

$$p_j = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{5} (1 - \frac{r_j}{r_i})} \{1 + b_1 r_j^1 + b_2 r_j^2 + b_3 r_j^3 + b_4 r_j^4\}$$

$j = 1, 2, 3, 4, 5$

And

$$p_{j+5} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{5} (1 - \frac{r_j}{r_i})} \times \{28 + (668 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4) r_j + (-21 + 100a_1 - 260a_2 + 1920a_4) r_j^2 + (18 - 75a_1 + 240a_2 - 540a_3 + 720a_4) r_j^3\}$$

$$p_{j+10} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{5} (1 - \frac{r_j}{r_i})} \times \{8 + (-154 + 760a_1 - 2880a_2 + 7680a_3 - 11520a_4) r_j + (1 + 10a_1 - 100a_2 + 480a_3 - 1200a_4) r_j^2 + (-1 + 5a_1 - 20a_2 + 60a_3 - 120a_4) r_j^3\}$$

$$p_{j+15} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{5} (1 - \frac{r_j}{r_i})} \times \{4 + (322 - 1540a_1 + 570a_2 - 15360a_3 + 23040a_4) r_j + (23 - 130a_1 + 580a_2 - 1920a_3 + 3840a_4) r_j^2 + (3 - 15a_1 + 60a_2 - 180a_3 + 360a_4) r_j^3\}$$

$$p_{j+20} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{5} (1 - \frac{r_j}{r_i})} \times \{8 + (-158 + 760a_1 - 2880a_2 + 7680a_3 - 11520a_4) r_j + (-21 + 110a_1 - 460a_2 + 1440a_3 - 2640a_4) r_j^2 + (-3 + 15a_1 - 60a_2 + 180a_3 - 360a_4) r_j^3\}$$

$$p_{j+25} = \frac{1}{\prod_{\substack{i=1 \\ i \neq j}}^{5} (1 - \frac{r_j}{r_i})} \times \{28 + (640 - 3100a_1 + 11520a_2 - 30720a_3 + 46080a_4) r_j$$

$+(125-640a_1+2620a_2-7680a_3+$
$13440a4rj2$
$+(25-125a_1+500a_2-1500a_3+2640a_4)r_j^3$
$+(3-15a_1+60a_2-180a_3+360a_4)r_j^4\}$

Hence Eq. (16) becomes:

$$U(t+l)=\sum_{i=1}^{5}A_i^{-1}\left[p_iU(t)+\frac{l}{360}\right]$$
$$\left\{\begin{matrix} p_{i+5}v(t)+16p_{i+10}v\left(t+\frac{l}{4}\right)+12p_{i+15}v\left(t+\frac{l}{2}\right) \\ +16p_{i+20}v\left(t+\frac{3\,l}{4}\right)+p_{i+25}v(t+l) \end{matrix}\right\}$$

where,

$$A_i=I-\frac{l}{r_i}A_i\ , i=1,2,3,4,5$$

Hence,

$$\boldsymbol{U}(t+l)=\sum_{i=1}^{5}y_i(t)$$

where, $y_i\ , i=1,2,3,4,5$ are the solutions of the systems:

$$\boldsymbol{A}_iy_i=p_i\boldsymbol{U}(t)+\frac{l}{360}$$
$$\left\{\begin{matrix} p_{i+5}\boldsymbol{v}(t)+16p_{i+10}\boldsymbol{v}\left(t+\frac{l}{4}\right)+12p_{i+15}\boldsymbol{v}\left(t+\frac{l}{2}\right) \\ +16p_{i+20}\boldsymbol{v}\left(t+\frac{3\,l}{4}\right)+p_{i+25}\boldsymbol{v}(t+l) \end{matrix}\right\}$$

NUMERICAL EXAMPLES

Numerical method described in this study will be applied to four problems from the literature and results obtained will be compared with exact solutions as well as with the results existing in the literature. We select values of $a_i(i=1,2,3,4)$ such that stability conditions are satisfied (Rehman *et al*., 2012).

Example 1: Consider the problem (1)-(4) with:

$F(x)=x^2, 0<x<1,$
$\varphi(x,t)=x, 0<x<1,$
$\tau(x,t)=x, 0<x<1,$
$G_1(t)=\dfrac{-1}{4(t+1)^2}, 0<t<1,$
$G_2(t)=\dfrac{3}{4(t+1)^2}, 0<t<1,$
$Q(x,t)=\dfrac{-2(x^2+t+1)}{(t+1)^3}, 0<x<1, 0<t\le 1$

which has the theoretical solution $u(x,t)=\left(\frac{x}{t+1}\right)^2$

For the comparison purpose the problem is solved for $\mathrm{h}=0.05, 0.025, 0.01, 0.005, 0.0025$. The relative errors obtained by new scheme are given in Table 1 and are compared with Crandall method, FTCS scheme, Dufort-Frankel scheme (Deghan, 2003) and fourth order scheme (Rehman and Taj, 2009).

Example 2: Consider the problem (1)-(4) with:

$F(x)=\exp(-x)\ , 0<x<1,$
$\varphi(x,t)=ax, 0<x<1,$
$\tau(x,t)=bx\cos(x)\ , 0<x<1,$
$G_1(t)=0, 0<t<1,$
$G_2(t)=0, 0<t<1,$
$Q(x,t)=-\exp[-\exp(x+\sin t)]\,(1+$
$\cos t), 0<x<1, 0<t\le 1$

where, $a=\frac{e}{e-2}$ and $b=\frac{2}{\sin(1)-\cos(1)+exp(1)}$ which has the theoretical solution $u(x,t)=\exp(-(x+\sin t))$

For the comparison purpose the problem is solved for $\mathrm{h}=0.05, 0.025, 0.01, 0.005, 0.0025$. The relative errors obtained by new scheme are given in Table 2 and are compared with Crandall method, FTCS scheme, Dufort-Frankel scheme (Deghan, 2003) and fourth order scheme (Rehman and Taj, 2009).

Example 3: Consider the problem (1)-(4) with:

Table 1: Comparison of relative error for $t=1$

Spatial length	Crandall	FTCS	Dufort-frankel	Fourth order method	New scheme
$\mathrm{h}=0.0500$	3.8×10^{-03}	7.5×10^{-02}	7.8×10^{-02}	2.6×10^{-06}	2.7×10^{-08}
$\mathrm{h}=0.0250$	2.1×10^{-04}	1.9×10^{-02}	1.9×10^{-02}	2.1×10^{-07}	1.1×10^{-09}
$\mathrm{h}=0.0100$	1.2×10^{-05}	4.0×10^{-03}	3.9×10^{-03}	6.1×10^{-09}	7.1×10^{-12}
$\mathrm{h}=0.0050$	7.1×10^{-07}	1.0×10^{-03}	1.0×10^{-03}	3.5×10^{-10}	4.4×10^{-11}
$\mathrm{h}=0.0025$	4.3×10^{-08}	2.5×10^{-04}	2.4×10^{-04}	8.0×10^{-11}	2.4×10^{-11}

Table 2: Comparison of relative error for $t=0.1$

Spatial length	Crandall	FTCS	Dufort-frankel	Fourth order method	New scheme
$\mathrm{h}=0.0500$	3.9×10^{-03}	6.4×10^{-02}	6.8×10^{-02}	3.0×10^{-07}	5.6×10^{-09}
$\mathrm{h}=0.0250$	2.4×10^{-04}	1.6×10^{-02}	1.7×10^{-02}	1.9×10^{-08}	3.7×10^{-09}
$\mathrm{h}=0.0100$	1.5×10^{-05}	4.1×10^{-03}	4.1×10^{-03}	5.0×10^{-10}	9.6×10^{-11}
$\mathrm{h}=0.0050$	1.0×10^{-06}	1.0×10^{-03}	1.0×10^{-03}	7.9×10^{-12}	5.2×10^{-12}
$\mathrm{h}=0.0025$	6.4×10^{-08}	2.5×10^{-04}	2.6×10^{-04}	7.0×10^{-11}	3.3×10^{-12}

Table 3: Comparison of absolute error for $h = l = 0.01$

t	Exact u	Crank-nicolson	The implicit	Fourth order method	New scheme
0.1	1.2796	5.2×10^{-05}	4.3×10^{-05}	4.8×10^{-06}	2.0×10^{-12}
0.2	1.1578	6.2×10^{-05}	6.0×10^{-05}	4.7×10^{-06}	2.4×10^{-12}
0.3	1.0476	6.5×10^{-05}	6.4×10^{-05}	3.9×10^{-06}	2.5×10^{-12}
0.4	0.9479	6.4×10^{-05}	6.3×10^{-05}	4.8×10^{-06}	2.4×10^{-12}
0.5	0.8577	6.2×10^{-05}	5.9×10^{-05}	5.3×10^{-06}	2.2×10^{-12}
0.6	0.7761	5.6×10^{-05}	4.8×10^{-05}	3.7×10^{-06}	2.0×10^{-12}
0.7	0.7022	5.0×10^{-05}	4.9×10^{-05}	2.3×10^{-06}	1.8×10^{-12}
0.8	0.6354	1.6×10^{-05}	1.5×10^{-05}	1.6×10^{-06}	1.6×10^{-12}
0.9	0.5749	4.1×10^{-05}	3.3×10^{-05}	1.1×10^{-06}	1.5×10^{-12}
1.0	0.5202	5.0×10^{-05}	4.7×10^{-05}	1.0×10^{-06}	1.3×10^{-12}

Table 4: Comparison of absolute error for $h = l = 0.01$

t	Fourth order method	New scheme
0.1	1.09×10^{-12}	5.34×10^{-14}
0.2	1.35×10^{-12}	6.58×10^{-14}
0.3	1.35×10^{-12}	6.71×10^{-14}
0.4	1.28×10^{-12}	6.16×10^{-14}
0.5	1.18×10^{-12}	5.72×10^{-14}
0.6	1.07×10^{-12}	5.25×10^{-14}
0.7	9.71×10^{-13}	4.85×10^{-14}
0.8	8.80×10^{-13}	4.41×10^{-14}
0.9	7.69×10^{-13}	3.88×10^{-14}
1.0	7.20×10^{-13}	3.59×10^{-14}

$F(x) = \sin(\pi x) + \cos(\pi x)\,, 0 < x < 1,$
$\varphi(x,t) = 2\sin(\pi x)\,, 0 < x < 1,$
$\tau(x,t) = -2\cos(\pi x)\,, 0 < x < 1,$
$G_1(t) = 0, 0 < t < 1,$
$G_2(t) = 0, 0 < t < 1,$
$Q(x,t) = (\pi^2 - 1)\exp(-t)\,\{\sin(\pi x) +$
$\cos \pi x, 0 < x < 1, 0 < t \leq 1$

which has the theoretical solution:

$$u(x,t) = \exp(-t)\,\{\sin(\pi x) + \cos(\pi x)\}$$

For the comparison purpose, in this problem is solved for $h = l = 0.01$ for different values of t. The errors obtained by new scheme are given in Table 3 and compared with fourth order scheme (Rehman and Taj, 2009).

Example 4: Consider the problem (1)-(4) with:

$F(x) = x(x-1) + \frac{D}{6(1+D)}, 0 < x < 1,$
$\varphi(x,t) = -D, 0 < x < 1,$
$\tau(x,t) = -D, 0 < x < 1,$
$G_1(t) = 0, 0 < t < 1,$
$G_2(t) = 0, 0 < t < 1,$
$Q(x,t) = \left[x(x-1) + \frac{D}{6(1+D)}\right]\exp(-t)\ , D =$
$0.0144\ 0 < x < 1, 0 < t \leq 1$

which has the theoretical solution:

$$u(x,t) = [x(x-1) + D/(6(1+D))]exp(-t)$$

For the comparison purpose the problem is solved for $h = l = 0.01$. The errors obtained by new scheme are given in Table 4 and compared with fourth order scheme (Rehman and Taj, 2009).

CONCLUSION

It is observed that the result obtained using hybrid scheme are highly precise in space and time. This technique can be coded easily on serial and parallel computers. The method involve only real domain and multiprocessor design, especially in nonlocal problems save significant computational time rather than the complex arithmetic based methods. This method is very flexible, user friendly and can be extended for multidimensional partial differential equations.

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