

# On Fuzzy Ideals in Rings and Anti-homomorphism

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**Abstract.** We investigate anti-homomorphic images and pre images of semiprime, strongly primary, irreducible and strongly irreducible fuzzy ideals of a ring. We also prove that: For a surjective anti-homomorphism  $f : R \rightarrow R'$ , if every fuzzy ideal of  $R$  is  $f$ -invariant and has a fuzzy primary (respectively, strongly primary) decomposition in  $R$ , then every fuzzy ideal of  $R'$  has a fuzzy primary (respectively, strongly primary) decomposition in  $R'$ .

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## 1. INTRODUCTION

The theory of fuzzy sets, proposed by Zadeh [20], has provided a useful mathematical device for unfolding the behavior of the systems that are too complex or nonspecific to admit precise mathematical analysis by conventional methods and tools. The study of fuzzy algebraic structure has started by Rosenfeld [16] and since then this concept has been applied to a variety of algebraic structures. Liu introduced the concept of a fuzzy ideal of a ring in [7]. The concepts of prime fuzzy ideals, maximal fuzzy ideals, primary fuzzy ideals were introduced in [9], [10] and [11]. Also, Malik [8], Mukerjee and

Sen [13], studied rings with chain conditions with the help of fuzzy ideals. The notion of fuzzy quotient ring was introduced by Kumar [4], Kuroaka and Kuroki [5]. In [6], K. H. Lee examined some properties of fuzzy quotient rings and used them to characterize Artinian and Noetherian rings. Recently in [17], we introduced strongly primary fuzzy ideals and strongly irreducible fuzzy ideals of a ring. We have discussed fuzzy quotient rings to characterize Laskerian rings.

In [18], Sheikabdullah and Jeyaraman has discussed anti- homomorphic images and pre images of prime fuzzy ideals and anti-homomorphic image of primary fuzzy ideals in a ring.

In this note we investigate anti- homomorphic images and pre images of semiprime, strongly primary, irreducible and strongly irreducible fuzzy ideals. We also prove that: for a surjective anti-homomorphism  $f : R \rightarrow R'$ , if every fuzzy ideal of  $R$  is  $f$ -invariant and has a fuzzy primary (respectively, strongly primary) decomposition in  $R$ , then every fuzzy ideal of  $R'$  has a fuzzy primary (respectively, strongly primary) decomposition in  $R'$ .<sup>1</sup>

## 2. PRELIMINARIES

In this section, review of fuzzy set theoretic concepts are given briefly. Following [20], a fuzzy subset of a non-empty set  $X$  is a function  $\mu : X \rightarrow [0, 1]$ . By [19], the set of all fuzzy subsets of a set  $X$  with the relation  $\mu \subseteq \nu$  if  $\mu(x) \leq \nu(x), \forall x \in X$  is a complete lattice. Whereas for a non empty family  $\{\mu_i : i \in I\}$  of fuzzy subsets of  $X$ ,

$$\inf\{\mu_i : i \in I\} : X \rightarrow [0, 1], x \mapsto \inf\{\mu_i(x) : i \in I\} \text{ and} \\ \sup\{\mu_i : i \in I\} : X \rightarrow [0, 1], x \mapsto \sup\{\mu_i(x) : i \in I\}, \forall x \in X.$$

A fuzzy subset  $\mu$  of a ring  $R$  is called a fuzzy subring of  $R$  if for all  $x, y \in R$ ,

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \text{ and } \mu(xy) \geq \min\{\mu(x), \mu(y)\}.$$

and a fuzzy subset  $\mu$  of a ring  $R$  is called a fuzzy ideal of  $R$  if for all  $x, y \in R$ ,

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \text{ and } \mu(xy) \geq \max\{\mu(x), \mu(y)\}.$$

If  $\mu$  is a fuzzy subset of  $R$ , then for  $t \in [0, 1]$  the set  $\mu_t = \{x \in R : \mu(x) \geq t\}$  is called a level subset of  $R$  with respect to  $\mu$ . A fuzzy subset is a fuzzy left ideal if and only if  $\mu(0) \geq \mu(x) \forall x \in R$  and  $\mu_t$  is a left ideal of  $R, \forall t \in [0, \mu(0)]$ . We denote  $\mu_* = \{x \in R : \mu(x) = \mu(0)\}$ . Following [18], a fuzzy ideal  $\mu$  of a ring  $R$  is called a prime fuzzy ideal if for any two fuzzy ideals  $\sigma$  and  $\theta$  of  $R$  the condition  $\sigma\theta \subseteq \mu$  implies that  $\sigma \subseteq \mu$  or  $\theta \subseteq \mu$ . Following [10], [11] for a fuzzy ideal  $\mu$  of a ring  $R$ , the fuzzy radical of  $\mu$ , denoted by  $\sqrt{\mu}$ , is defined by  $\sqrt{\mu} = \cap\{\sigma : \sigma \text{ is a fuzzy prime ideal of } R, \sigma \subseteq \mu, \sigma_* \subseteq \mu_*\}$ . In [15], V.

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Murali and B. Makaba has discussed concepts of primary decomposition of fuzzy ideals and the radicals of such ideals over a Noetherian ring.

The operations of intersection, sum and product on fuzzy subsets  $\mu$  and  $\nu$  of a ring  $R$  are defined as follows:

$$\begin{aligned} (\mu \cap \nu)(x) &= \inf\{\mu(x), \nu(x)\} \\ (\mu + \nu)(x) &= \sup\{\mu(r)\wedge\nu(s) : x = r + s \in R\} \\ (\mu\nu)(x) &= \sup\{\mu(r)\wedge\nu(s) : x = rs, r, s \in R\}, \\ (\mu\nu)(x) &= \sup\{\wedge_{i=1}^n \mu(r_i)\wedge\nu(s_i) : x = \sum_{i=1}^n r_i s_i, r_i, s_i \in R, n \in \mathbb{N}\}. \end{aligned}$$

If  $x$  has no such decomposition, the sum and product take the value 0 at  $x$ .

Recall that a ring  $R$  is Laskerian if each ideal of  $R$  admits a shortest primary representation;  $R$  is strongly Laskerian, if  $R$  is Laskerian and each primary ideal of  $R$  contains a power of its radical [2, Page 455]. It is equivalent to say that a ring  $R$  is Laskerian (resp. strongly Laskerian) if every ideal of  $R$  can be represented as a finite intersection of primary ideals (resp. strongly primary ideals)[1, Page 295, 298]. It is well known that

$$\text{Noetherian} \implies \text{Strongly Laskerian} \implies \text{Laskerian}$$

### 3. MAIN RESULTS

Following [18], a fuzzy ideal  $\mu$  of a ring  $R$  is known as a primary fuzzy ideal if for any two fuzzy ideals  $\sigma$  and  $\theta$  of  $R$  the conditions  $\sigma\theta \subseteq \mu$  and  $\sigma \not\subseteq \mu$  together imply that  $\theta \subseteq \sqrt{\mu}$ .

**Lemma 1.** *If  $\mu$  is a primary fuzzy ideal of  $R$ , then  $\sqrt{\mu}$  is a prime fuzzy ideal of  $R$ .*

*Proof.* Let  $\sigma$  and  $\theta$  be two fuzzy ideal of  $R$  such that  $\sigma\theta \subseteq \sqrt{\mu}$  and  $\sigma \not\subseteq \sqrt{\mu}$ . Then  $\sigma \not\subseteq \mu$  and so  $\theta \subseteq \sqrt{\mu}$ . Hence  $\sqrt{\mu}$  is a prime fuzzy ideal of  $R$ .  $\square$

**Remark 1.** In a Laskerian ring an irreducible fuzzy ideal is primary (cf [17, Proposition 6]), therefore if  $\mu$  is an irreducible fuzzy ideal of a Laskerian ring, then  $\sqrt{\mu}$  is a prime fuzzy ideal.

Let  $X$  and  $Y$  be two non empty sets,  $f : X \rightarrow Y$ ,  $\mu$  and  $\sigma$  be fuzzy subsets of  $X$  and  $Y$  respectively. Then  $f(\mu)$ , the image of  $\mu$  under  $f$  is a fuzzy subset of  $Y$  denoted by  $(f(\mu))(y) = \begin{cases} \sup_{f(x)=y} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$

$f^{-1}(\sigma)$ , the pre image of  $\sigma$  under  $f$  is a fuzzy subset of  $X$  defined by  $(f^{-1}(\sigma))(x) = \sigma(f(x))$  for all  $x \in X$ .

Following [18, Definition 2.8], a mapping  $f : R \rightarrow R'$  is called a fuzzy anti-homomorphism of rings, if

$$f(\mu + \sigma) = f(\mu) + f(\sigma) \text{ and } f(\mu\sigma) = f(\sigma)f(\mu).$$

where  $\mu$  and  $\sigma$  are fuzzy ideals of  $R$ .

**Proposition 1.** *Let  $f : R \rightarrow R'$  be a surjective ring anti-homomorphism, if  $\mu'$  is a fuzzy primary ideal of  $R'$ , then  $f^{-1}(\mu')$  is a primary fuzzy ideal of  $R$ .*

*Proof.* Let  $\theta$  and  $\sigma$  be any two fuzzy ideals of  $R$ , such that  $\theta\sigma \subset f^{-1}(\mu')$ . This implies that  $f(\theta\sigma) \subset ff^{-1}(\mu') = \mu'$  which means  $f(\sigma)f(\theta) \subset \mu'$ . Because  $\mu'$  is a primary fuzzy ideal of  $R'$  therefore,  $f(\sigma) \subset \mu'$  or  $f(\theta) \subset \sqrt{\mu'}$ . So  $f^{-1}(f(\sigma)) \subset f^{-1}(\mu')$  or  $f^{-1}(f(\theta)) \subset f^{-1}(\sqrt{\mu'})$ . That is,  $\sigma \subset f^{-1}(\mu')$  or  $\theta \subset f^{-1}(\sqrt{\mu'}) = \sqrt{f^{-1}(\mu')}$ . As a result  $f^{-1}(\mu')$  is a primary fuzzy ideal of  $R'$ .  $\square$

Following [17], a fuzzy ideal  $\mu$  of a ring  $R$  is called strongly primary fuzzy ideal of  $R$  if  $\mu$  is a primary fuzzy ideal and  $(\sqrt{\mu})^n \subset \mu$  for some  $n \in \mathbb{N}$ .

For a function  $f : R \rightarrow R'$ , a fuzzy subset  $\mu$  of a ring  $R$  is called  $f$ -invariant if  $f(x) = f(y)$  implies  $\mu(x) = \mu(y)$ ,  $x, y \in R$ . Clearly, if  $\mu$  is  $f$ -invariant then  $f^{-1}(f(\mu)) = \mu$ .

**Proposition 2.** *For a surjective ring anti-homomorphism (homomorphism)  $f : R \rightarrow R'$ , if  $\mu$  is an  $f$ -invariant strongly primary fuzzy ideal of  $R$ , then  $f(\mu)$  is a strongly primary fuzzy ideal of  $R'$ .*

*Proof.* In [18, Proposition 3.5], it is proved that  $f(\mu)$  is a primary fuzzy ideal of  $R'$ . We need to prove that  $[\sqrt{f(\mu)}]^n \subset f(\mu)$  for some  $n \in \mathbb{N}$ . By Lemma [1],  $\sqrt{f(\mu)}$  is a prime fuzzy ideal of  $R'$ , therefore  $[\sqrt{f(\mu)}]^n \subset f(\mu)$  for some  $n \in \mathbb{N}$ . Hence  $f(\mu)$  is a strongly primary ideal of  $R'$ .  $\square$

Following [3], a fuzzy ideal  $\mu$  of a ring  $R$  is called semiprime if  $\mu^2(x) = \mu(x)$  for all  $x \in R$ .

**Proposition 3.** *For a surjective ring anti-homomorphism (homomorphism)  $f : R \rightarrow R'$ , if  $\mu'$  is a semiprime fuzzy ideal of  $R'$ , then  $f^{-1}(\mu')$  is a semiprime fuzzy ideal of  $R$ .*

*Proof.* If  $\mu'$  is a semiprime fuzzy ideal of  $R'$ , then by [18, Proposition 3.2],  $f^{-1}(\mu')$  is a fuzzy ideal of  $R$ . We only need to show that  $[f^{-1}(\mu')]^2 = f^{-1}(\mu')$ . For this let  $f^{-1}(\mu') = \mu$  this implies that  $\mu' = f(\mu)$ . Now  $\mu' = \mu'\mu' = f(\mu)f(\mu) = f(\mu\mu) = f(\mu^2) \Rightarrow \mu^2 = f^{-1}(\mu') = \mu$ . Hence proved.  $\square$

**Proposition 4.** *Let  $f : R \rightarrow R'$  be a ring anti-homomorphism (homomorphism). If  $\mu$  is any  $f$ -invariant semiprime fuzzy ideal of  $R$ , then  $f(\mu)$  is a semiprime fuzzy ideal of  $R'$ .*

*Proof.* If  $\mu$  is an  $f$ -invariant semiprime fuzzy ideal of  $R$ , then by [18, Proposition 3.1],  $f(\mu)$  is a fuzzy ideal of  $R'$ . We only need to show that  $[f(\mu)]^2 = f(\mu)$ . For this consider

$$[f(\mu)]^2 = f(\mu)f(\mu) = f(\mu^2) = f(\mu). \quad \square$$

Following [15, Definition 4.1], a fuzzy ideal  $\mu$  of a Noetherian ring  $R$  is said to be irreducible if  $\mu \neq R$  and whenever  $\mu = \mu_1 \wedge \mu_2$ , where  $\mu_1, \mu_2$  are fuzzy ideals of  $R$ , then  $\mu = \mu_1$  or  $\mu = \mu_2$ . A prime fuzzy ideal is irreducible;

In the rest of text we consider ring to be a Laskerian ring.

Following [17], a proper fuzzy ideal  $\mu$  of a ring  $R$  is said to be strongly irreducible if for each pair of fuzzy ideals  $\theta$  and  $\sigma$  of  $R$ , if  $\theta \wedge \sigma \subseteq \mu$ , then either  $\theta \subseteq \mu$  or  $\sigma \subseteq \mu$ .

**Remark 2.** *A strongly irreducible fuzzy ideal is irreducible.*

**Proposition 5.** *Let  $f : R \rightarrow R'$  be a surjective ring anti-homomorphism. If  $\mu'$  is an irreducible fuzzy ideal of  $R'$ , then  $f^{-1}(\mu')$  is an irreducible fuzzy ideal of  $R$ .*

*Proof.* Let  $\mu$  and  $\sigma$  be any two fuzzy ideals of  $R$ , such that  $\mu \wedge \sigma = f^{-1}(\mu')$ . This implies that  $f(\mu \wedge \sigma) = f f^{-1}(\mu') = \mu'$ . Because  $f$  is a surjective anti-homomorphism, therefore  $f(\sigma) \wedge f(\mu) = \mu'$ . As  $\mu'$  is irreducible fuzzy ideal of  $R'$  therefore  $f(\sigma) = \mu'$  or  $f(\mu) = \mu'$ . So  $f^{-1}(f(\sigma)) = f^{-1}(\mu)$  or  $f^{-1}(f(\mu)) = f^{-1}(\mu')$ , this means  $\sigma = f^{-1}(\mu')$  or  $\mu = f^{-1}(\mu')$ . Hence  $f^{-1}(\mu')$  is a fuzzy irreducible ideal of  $R'$ .  $\square$

**Proposition 6.** *Let  $f : R \rightarrow R'$  be a surjective ring anti-homomorphism. If  $\mu'$  is a strongly irreducible fuzzy ideal of  $R'$ , then  $f^{-1}(\mu')$  is a strongly irreducible fuzzy ideal of  $R$ .*

*Proof.* Let  $\mu$  and  $\sigma$  be any two fuzzy ideals of  $R$ , such that  $\mu \wedge \sigma \subseteq f^{-1}(\mu')$ . This implies that  $f(\mu \wedge \sigma) \subseteq f f^{-1}(\mu') = \mu'$ . Because  $f$  is a surjective anti-homomorphism  $f(\sigma) \wedge f(\mu) \subseteq \mu'$ . This means that  $\inf\{f(\sigma), f(\mu)\} \subseteq \mu'$  so,  $f(\sigma) \subseteq \mu'$  or  $f(\mu) \subseteq \mu'$ . Therefore  $f^{-1}(f(\sigma)) \subseteq f^{-1}(\mu')$  or  $f^{-1}(f(\mu)) \subseteq f^{-1}(\mu')$ , consequently  $\sigma \subseteq f^{-1}(\mu')$  or  $\mu \subseteq f^{-1}(\mu')$ . Hence  $f^{-1}(\mu')$  is a strongly irreducible fuzzy ideal of  $R'$ .  $\square$

**Proposition 7.** *For a ring anti-homomorphism  $f : R \rightarrow R'$ , if  $\mu$  is an  $f$ -invariant strongly irreducible fuzzy ideal of  $R$ , then  $f(\mu)$  is a strongly irreducible fuzzy ideal of  $R'$ .*

*Proof.* Let  $\sigma'$  and  $\theta'$  be any two fuzzy ideals of  $R$ , such that  $\sigma' \wedge \theta' \subseteq f(\mu)$ . This implies that  $f^{-1}(\sigma' \wedge \theta') \subseteq f^{-1}f(\mu) = \mu$ . therefore  $f^{-1}(\theta') \wedge f^{-1}(\sigma') \subseteq \mu$ . Since  $\mu$  is strongly irreducible fuzzy ideal of  $R$  therefore  $f^{-1}(\theta') \subseteq \mu$  or  $f^{-1}(\sigma') \subseteq \mu$ . So  $f f^{-1}(\theta') \subseteq f(\mu)$  or  $f f^{-1}(\sigma') \subseteq f(\mu)$  consequently  $\theta' \subseteq f(\mu)$  or  $\sigma' \subseteq f(\mu)$  that is  $f(\mu)$  is strongly irreducible fuzzy ideal of  $R'$ .  $\square$

**Proposition 8.** *Let  $f : R \rightarrow R'$  be a ring anti-homomorphism. If  $\mu$  is an  $f$ -invariant irreducible fuzzy ideal of  $R$ , then  $f(\mu)$  is an irreducible fuzzy ideal of  $R'$ .*

*Proof.* Let  $\sigma'$  and  $\theta'$  be any two fuzzy ideals of  $R$ , such that  $\sigma' \wedge \theta' = f(\mu)$ . This implies that  $f^{-1}(\sigma' \wedge \theta') = f^{-1}f(\mu) = \mu = f^{-1}(\theta') \wedge f^{-1}(\sigma')$ . Since  $\mu$  is

irreducible fuzzy ideal of  $R$ , therefore, either  $f^{-1}(\theta') = \mu$  or  $f^{-1}(\sigma') = \mu$ . So  $ff^{-1}(\theta') = f(\mu)$  or  $ff^{-1}(\sigma') = f(\mu)$ . This means  $\theta' = f(\mu)$  or  $\sigma' = f(\mu)$ . Hence  $f(\mu)$  is an irreducible fuzzy ideal of  $R'$ .  $\square$

Following [15, Definition 3.1 and 3.2], if for a collection  $\{v_i : i = 1, 2, \dots, n\}$  of primary fuzzy ideals of  $R$  and  $\{\mu_i : i = 1, 2, \dots, n\}$  a finite collection of  $v_i$ -primary fuzzy ideals of  $R$ , then,  $\mu = \Lambda_{i=1}^n \mu_i$  is called a primary decomposition of  $\mu$ . This decomposition is said to be reduced or irredundant if

- (a) the  $v_1, v_2, \dots, v_n$  are all distinct and
- (b)  $\mu_j \not\subseteq \Lambda_{i=1, i \neq j}^n \mu_i$ , for all  $j = 1, 2, \dots, n$ .

**Proposition 9.** *For a surjective ring anti-homomorphism  $f : R \rightarrow R'$ , if every fuzzy ideal of  $R$  is  $f$ -invariant and has a fuzzy primary (respectively, strongly primary) decomposition in  $R$ , then every fuzzy ideal of  $R'$  has a fuzzy primary (respectively, strongly primary) decomposition in  $R'$ .*

*Proof.* If  $\mu'$  is a fuzzy ideal of  $R'$ , then  $f^{-1}(\mu')$  is a fuzzy ideal of  $R$ . Then  $f^{-1}(\mu') = \Lambda_{i=1}^n \mu_i$  where  $\mu_i$  are primary (respectively, strongly primary) fuzzy ideals of  $R$  and  $i = 1, 2, \dots, n$ . This implies that  $ff^{-1}(\mu') = f(\Lambda_{i=1}^n \mu_i)$  that is,  $\mu' = \Lambda_{i=1}^n f(\mu_i)$  by [18, Proposition 3.5],  $f(\mu_i)$  are primary (respectively, strongly primary (by Proposition 2)) fuzzy ideals for all  $i = 1, 2, \dots, n$ .

Hence  $\mu'$  has a fuzzy primary (respectively, strongly primary) decomposition in  $R'$ .  $\square$

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