SPANNING SIMPLICIAL COMPLEXES OF UNI-CYCLIC GRAPHS

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ABSTRACT. In this paper, we introduce the concept of *spanning simplicial complexes* $\Delta_s(G)$ associated to a simple finite connected graph G. We give the characterization of all spanning trees of the *uni-cyclic graph* $U_{n,m}$. In particular, we give the formula for computing the Hilbert series and h-vector of the Stanley Riesner ring $k\big[\Delta_s(U_{n,m})\big]$. Finally, we prove that the *spanning simplicial complex* $\Delta_s(U_{n,m})$ is shifted hence $\Delta_s(U_{n,m})$ is shellable.
Key words : Primary Decomposition, Hilbert Series, f-vectors, h-vectors, spanning Trees.
2000 Mathematics Subject Classification: Primary 13P10, Secondary 13H10, 13F20, 13C14.

1. INTRODUCTION

Suppose $G(V,E)$ is a finite simple connected graph with the vertex set V and edge-set E, a spanning tree of a simple connected finite graph G is a subgraph of G that contains every vertex of G and is also a tree. We represent the edge-set of all spanning trees of a graph G by $s(G)$. In this paper, for a finite simple connected graph $G(V,E)$, we introduce the concept of *spanning simplicial complexes* by associating a simplicial complex $\Delta_s(G)$ defined on the edge set E of the graph G as follows:

$$\Delta_s(G) = \big\langle F_i \mid F_i \in s(G)\big\rangle$$

It is always possible to associate $\Delta_s(G)$ to any simple finite connected graph $G(V,E)$ but the characterization of $s(G)$ has been a problem in this regard.

For the *uni-cyclic graphs* $U_{n,m}$, we prove some algebraic and combinatorial properties of *spanning simplicial complex* $\Delta_s(U_{n,m})$. Where, a uni-cyclic graph $U_{n,m}$ is a connected graph over n vertices and containing exactly one cycle of length m. In Proposition 3.1, we give the characterization of $s(U_{n,m})$. Moreover, we give characterizations of the $f-vector$ and $h-vector$ in Lemma 3.3 and Theorem 3.5 respectively, which enable us to device a formula to compute the *Hilbert series* of the Stanley Reisner ring $k\big[\Delta_s(U_{n,m})\big]$ in Theorem 3.6. In the Theorem 3.8, we show that the *spanning simplicial complex* $\Delta_s(U_{n,m})$ is *shifted*. So, we have the corollary 3.9 that the *spanning simplicial complex* $\Delta_s(U_{n,m})$ is *shellable*.

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2. Basic Setup

In this section, we give some basic definitions and notations which we will follow in this paper.

Definition 2.1. A spanning tree of a simple connected finite graph $G(V, E)$ is a subtree of G that contains every vertex of G.

We represent the collection of all edge-sets of the spanning trees of G by $s(G)$, in other words;

$$s(G) = \{E(T_i) \subset E, \text{ where } T_i \text{ is a spanning tree of } G\}.$$

Remark 2.2. It is well known that for any simple finite connected graph spanning tree always exist. One can find a spanning tree systematically by *cutting-down method*, which says that spanning tree of a given simple finite connected graph is obtained by removing one edge from each cycle appearing in the graph.

For example by using *cutting-down method* for the graph given in figure 1 we obtain:

$$s(G) = \{\{e_2, e_3, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3\}\}$$

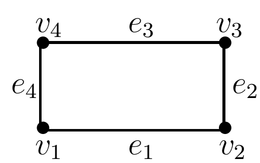

Fig. 1 . C_4

Definition 2.3. A Simplicial complex Δ over a finite set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, with the property that $\{i\} \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$ then Δ will contain all the subsets of F (including the empty set). An element of Δ is called a face of Δ, and the dimension of a face F of Δ is defined as $|F| - 1$, where $|F|$ is the number of vertices of F. The maximal faces of Δ under inclusion are called facets of Δ. The dimension of the simplicial complex Δ is :

$$\dim\Delta = \max\{\dim F | F \in \Delta\}.$$

We denote the simplicial complex Δ with facets $\{F_1, \ldots, F_q\}$ by

$$\Delta = \langle F_1, \ldots, F_q \rangle$$

Definition 2.4. For a simplicial complex Δ having dimension d, its $f-vector$ is a $d+1$-tuple, defined as:

$$f(\Delta) = (f_0, f_1, \ldots, f_d)$$

where f_i denotes the number of $i-dimensional$ faces of Δ.

Definition 2.5. (Spanning Simplicial Complex)
For a simple finite connected graph $G(V, E)$ with $s(G) = \{E_1, E_2, \ldots, E_s\}$ be the edge-set of all possible spanning trees of $G(V, E)$, we define a simplicial complex $\Delta_s(G)$ on E such that the facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, we call $\Delta_s(G)$ as the *spanning simplicial complex* of $G(V, E)$. In other words;

$$\Delta_s(G) = \langle E_1, E_2, \ldots, E_s \rangle.$$

For example; the spanning simplicial complex of the graph G given in figure 1 is:

$$\Delta_s(G) = \big\langle \{e_2, e_3, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3\} \big\rangle$$

We conclude this section with the definition of *uni-cyclic graph* $U_{n,m}$;

Definition 2.6. A *uni-cyclic graph* $U_{n,m}$ is a connected graph on n vertices, and containing exactly one cycle of length m (with $m \leq n$).

The number of vertices in $U_{n,m}$ equals the number of edges. In particular, if $m = n$ then $U_{n,m}$ is simply n-cyclic graph.

3. Spanning trees of $U_{n,m}$ and Stanley-Reisner ring $\Delta_s(U_{n,m})$

Throughout the paper, we fix the edge-labeling $\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ of $U_{n,m}$ such that $\{e_1, e_2, \ldots, e_m\}$ is the edge-set of the only cycle in $U_{n,m}$. In the following result, we give the characterization of $s(U_{n,m})$.

Lemma 3.1. Characterization of $s(U_{n,m})$
Let $U_{n,m}$ be the *uni-cyclic graph* with the edge set $E = \{e_1, e_2, \ldots, e_n\}$. A subset $E(T_i) \subset E$ will belong to $s(U_{n,m})$ if and only if $T_i = E \backslash \{e_i\}$ for some $i \in \{1, \ldots, m\}$. In particular;

$$s(U_{n,m}) = \{\hat{E}_i \mid \hat{E}_i = E \setminus \{e_i\} \text{ for all } 1 \leq i \leq m\}$$

Proof. As $U_{n,m}$ contains only one cycle of m vertices, so its spanning trees will be obtained by just removing one edge from the cycle of $U_{n,m}$ follows from 2.2. Which implies that

$$s(U_{n,m}) = \{\hat{E}_i \mid \hat{E}_i = E \setminus \{e_i\} \text{ for all } 1 \leq i \leq m\}$$

□

We need the following elementary proposition in order to prove our next result.

Proposition 3.2. For a simplicial complex Δ over $[n]$ of dimension d, if $f_t = \binom{n}{t+1}$ for some $t \leq d$ then $f_i = \binom{n}{i+1}$ for all $0 \leq i < t$.

Proof. Suppose Δ be any simplicial complex over $[n]$ with dimension d having $f_t = \binom{n}{t+1}$ for some $t \leq d$. It implies that Δ will contain all the subset of $[n]$ with the cardinality $t+1$ (which is $f_t = \binom{n}{t+1}$), then it is sufficient to prove that Δ will contain every subset of $[n]$ with the cardinality $\mid i \mid$ with $i \leq t$. Let us take any arbitrary subset F of $[n]$ with $\mid F \mid < t+1$, then by adding more vertices to F we can extend F to $\tilde{F}$ with $\mid \tilde{F} \mid = t+1$, which is already in Δ therefore the assertion follows immediately from the definition of simplicial complex. Hence Δ will contain all the subsets of $[n]$ with the cardinality $\leq t$, that is

$$f_i = \binom{n}{i+1} \text{ for all } 0 \leq i < t.$$

□

Our next result is the characterization of the f-vector of $\Delta_s(U_{n,m})$.

Proposition 3.3. Let $\Delta_s(U_{n,m})$ be the spanning simplicial complex of *uni-cyclic graph* $U_{n,m}$, then $dim(\Delta_s(U_{n,m}) = n-2$ and having the following f-vector $f(\Delta_s(U_{n,m}) = \big(f_0, f_1, \ldots, f_{n-2}\big)$ with

$$f_i = \begin{cases} \binom{n}{i+1}, & \text{for } i \le m-2; \\ \\ \binom{n}{i+1} - \binom{n-m}{i-m+1}, & \text{for } m-2 < i \le n-2. \end{cases}$$

Proof. Let $E = \{e_1, e_2, \ldots, e_n\}$ be the set of edges of $U_{n,m}$, then from 3.1 ;

$$s(U_{n,m}) = \{\hat{E_i} \mid E_i = E \setminus \{e_i\} \text{ for all } 1 \le i \le m\}.$$

Therefore, by definition 2.5 we have;

$$\Delta_s(U_{n,m}) = \big\langle \hat{E_1}, \hat{E_2}, \ldots, \hat{E_m} \big\rangle.$$

Since each facet $\hat{E_i}$ is of the same dimension $n-2$ (as $\mid \hat{E_i} \mid = n-1$), therefore $\Delta_s(U_{n,m})$ will be of dimension $n-2$. Also, it is clear from the definition of $\Delta_s(U_{n,m})$ that $\Delta_s(U_{n,m})$ contains all those subsets of E that do not contain $\{e_1, \ldots, e_m\}$.

Let us take any arbitrary subset $F \subset E$ consisting of $m-1$ members. As F consists of $m-1$ elements, then it is clear that $\{e_1, \ldots, e_m\}$ can not appear in F, therefor, $F \in \Delta_s(U_{n,m})$. It follows that $\Delta_s(U_{n,m})$ contains all possible subsets of E with the cardinality $m-1$, therefore, $f_{m-2} = \binom{n}{m-1}$. Hence from 3.2, we have $f_i = \binom{n}{i+1}$ for all $i \le m-2$.

In order to prove the other case, we need to compute all the subsets of $F \subset E$ with $\mid F \mid = i(\ge m)$ containing the cycle $\{e_1, \ldots, e_m\}$. We have in total n elements in E and we are choosing i-elements out of it with the condition that $\{e_1, \ldots, e_m\}$ will be a part of it. By using the inclusion exclusion principle, we get that there are $\binom{n-m}{i-m+1}$ subsets of E consisting of $i+1\,(\ge m)$ elements and containing the cycle $\{e_1, \ldots, e_m\}$. In total, we have $\binom{n}{i+1}$ subsets of E with the cardinality $i+1$, therefore, we have the $f_i = \binom{n}{i+1} - \binom{n-m}{i-m+1}$ for $m-2 < i \le n-2$. ☐

For a simplicial complex Δ over $[n]$, one would associate to it the Stanley-Reisner ideal, that is, the monomial ideal $I_{\mathcal{N}}(\Delta)$ in $S = k[x_1, x_2, \ldots, x_n]$ generated by monomials corresponding to non-faces of this complex (here we are assigning one variable of the polynomial ring to each vertex of the complex). It is well known that the Stanley-Reisner ring $k[\Delta] = S/I_{\mathcal{N}}(\Delta)$ is a standard graded algebra. We refer the readers to [6] and [8] for more details about graded algebra A, the Hilbert function $H(A,t)$ and the Hilbert series $H_t(A)$ of a graded algebra.

Definition 3.4. Let A be a standard graded algebra and

$$h(t) = h_0 + h_1 t + \cdots + h_r t^r$$

the (unique) polynomial with integral coefficients such that $h(1) \neq 0$ and satisfying

$$H_t(A) = \frac{h(t)}{(1-t)^d}$$

where $d = \dim(A)$. The h-vector of A is defined by $h(A) = (h_0, \ldots, h_r)$.

Now we give the formula for the h-vector of $k\big[\Delta_s(U_{n,m})\big]$;

Theorem 3.5. If $\Delta_s(U_{n,m})$ is a spanning simplicial complex of the uni-cyclic graph $U_{n,m}$ and (h_i) is the h-vector of $k\big[\Delta_s(U_{n,m})\big]$, then $h_k = 0$ for $k > n-1$ and

$$h_k = \begin{cases} \sum\limits_{i=0}^{k}(-1)^{k-i}\binom{n-1-i}{k-i}\binom{n}{i}, & \text{for } k \leq m-1; \\ \sum\limits_{i=0}^{k}(-1)^{k-i}\binom{n-1-i}{k-i}\Big[\binom{n}{i}-\binom{n-m}{i-m}\Big], & \text{for } m-1 < k \leq n-1. \end{cases}$$

Proof. We know from [8] that, if Δ be any simplicial complex of dimension d and (h_i) be the h-vector of $k[\Delta]$, then $h_k = 0$ for $k > d+1$ and

$$h_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{n-1-i}{k-i} f_{i-1} \quad \text{for} \quad 0 \leq k \leq d+1$$

The result follows by substituting the values f_i's (from 3.3) in the above formula. □

One of our main results of this section is as follows;

Theorem 3.6. Let $\Delta_s(U_{n,m})$ be the spanning simplicial complex of $U_{n,m}$, then the Hilbert series of the Stanley-Reisner ring $k\big[\Delta_s(U_{n,m})\big]$ is given by,

$$H(k[\Delta_s(U_{n,m})], t) = \sum_{i=0}^{m-2}\frac{\binom{n}{i+1}t^{i+1}}{(1-t)^{i+1}} + \sum_{i=m-1}^{n-2}\frac{\big[\binom{n}{i+1}-\binom{n-m}{i-m+1}\big]t^{i+1}}{(1-t)^{i+1}} + 1.$$

Proof. We know from [8] that if Δ be any simplicial complex of dimension d with $f(\Delta) = (f_0, f_1, \ldots, f_d)$ be its f-vector, then the Hilbert series of Stanley-Reisner ring $k[\Delta]$ is given by

$$H(k[\Delta], t) = \sum_{i=0}^{d}\frac{f_i t^{i+1}}{(1-t)^{i+1}} + 1.$$

The result immediately follows by substituting the values of f_i's in the above formula from 3.3. □

Algebraic shifting theory was introduced by G. Kalai in [7], and it describes strong tolls to investigate one of the most interesting and powerful property of simplicial complexes.

Definition 3.7. A simplicial complex Δ on $[n]$ is *shifted* if, for $F \in \Delta$, $i \in F$ and $j \in [n]$ with $j > i$, one has $(F \setminus \{i\}) \cup \{j\} \in \Delta$.

Theorem 3.8. The spanning simplicial complex $\Delta_s(U(m,n))$ of the *uni-cyclic graph* is *shifted*.

Proof. From 2.5 and 3.1, we know that

$$\Delta_s(U_{n,m}) = \big\langle \hat{E_1}, \hat{E_2}, \ldots, \hat{E_m}\big\rangle.$$

It is sufficient to prove the shifted condition on the facets of the simplicial complex. For some facet $\hat{E_i} \in \Delta_s(U_{n,m})$ with $j \in \hat{E_i}$, we claim that

$(\hat{E_i} \setminus \{j\}) \cup \{k\} \in \{\hat{E_1}, \hat{E_2}, \ldots, \hat{E_m}\}$ with $k \notin \hat{E_i}$ and $j < k$. By the definition of $\hat{E_i}$, we have only one possibility for k that is $k = i$ and $j < i \leq m$, therefore, it is easy to see that $(\hat{E_i} \setminus \{j\}) \cup \{i\} = \hat{E_i}$ for all $j(< i) \in \hat{E_i}$. Hence $\Delta_s(U_{n,m})$ is a shifted simplicial complex. □

The above theorem immediately implies the following result;

Corollary 3.9. The spanning simplicial complex $\Delta_s(U(m, n))$ is shellable.

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