Transient and Steady State Solution of N-Dimensional Coupled Networks and Development of Equivalent Pi and T Matrix Networks with Distributed Parameters

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Abstract- **This paper proposes a new computer based method for transient and steady state solution of n-dimensional coupled transmission line networks or communication circuits with distributed parameters. In this method, a novel approach has been developed for formulating and computing the n-dimensional generalized ABCD parameter matrices, as well as for developing and solving the equivalent n-dimensional coupled T and Pi networks with distributed parameters. The proposed method uses Cayley-Hamilton's theorem to compute the hyperbolic N-dimensional generalized ABCD parameter matrices with finite terms which are fundamental to the solution and development of the equivalent N-dimensional T and Pi matrix networks. The square root function of the complex matrix [W] is also computed with finite terms. As a result, truncation of matrices is eliminated, and an improved closed from solution is achieved. The method is straight forward, computationally efficient, and neither it involves the use of eigenvector based modal transformations necessary for diagonalization of parameter matrices nor it requires the evaluation of infinite series of hyperbolic functions with n-dimensional matrices as their arguments, and is extremely useful in the steady state and transient analysis of n-dimensional, unbalanced, coupled systems with distributed parameters. The method is extremely useful in the fault analysis of n-dimensional unbalanced coupled systems with distributed parameters. To date no such method is reported in the literature.**

KEY WORDS

N-phase transmission lines, Fault analysis, Hyperbolic functions of N-dimensional matrices, ABCD parameter matrices, N-dimensional circuits with distributed parameters, Coupled matrix circuits, Equivalent T matrix circuit, Equivalent Pi matrix circuit, Cayley-Hamilton's theorem, Solution of large networks.

I. INTRODUCTION

An n-dimensional transmission line network or a communication circuit with distributed parameters constitutes a coupled and complex symmetric or asymmetric network. Such networks can be represented by a set of coupled non-singular square matrices which may not necessarily be diagonalizable.

In the study of transient and steady state analysis of n-dimensional coupled and unbalanced circuits with distributed parameters, often it becomes desirable to represent such networks as equivalent or cascaded series of Pi or T n-dimensional matrix networks. The solution and development of such equivalent N-dimensional T and Pi matrix networks require the computation of n-dimensional generalized ABCD parameter matrix tensors. A computer based method which utilizes Caley-Hamilton's theorem has been proposed in this paper to compute such ABCD parameter matrices. This paper considers the development of n-dimensional equivalent T and Pi matrix networks of n-dimensional symmetric or asymmetric coupled circuits with distributed parameters [1, 2, 3] as opposed to those of one –dimensional scalar circuits treated in standard text book [4,6]. The proposed method utilizes Cayley-Hamilton's theorem, is straight forward, computationally efficient, and neither it involves the use of eigenvector based modal transformations necessary for diagonalization of parameter matrices nor does it require the evaluation of infinite series of hyperbolic functions with n-dimensional matrices as their arguments. The method computes the square root function of the complex matrix $[W_2]$ together with the hyperbolic n-dimensional generalized ABCD parameter matrices with finite terms, and thereby eliminating the truncation of matrices and achieving an improved closed form solution. The method is extremely useful in the steady state and transient analysis of n-dimensional, unbalanced coupled systems with distributed parameters.
To date no such method exists in the literature.

II. EFFECT OF MATRIX TRUNCATION ON COMPUTER BASED COMPUTATIONAL ACCURACIES

In practice there are many physical problems such as the computation of n-dimensional coupled communications circuits or transmission line networks with distributed parameters, or a navigation, guidance, and flight control multiprocessing system, in which the set of control commands are computed from the set of a highly complex control laws. Most of these problems involve the computation of e^A matrix functions or other similar complex function. Many of these algorithms or computations may encounter numerical difficulties, often severe, when pushed (e.g., on larger order

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matrices and relatively ill-conditioned data) due to the number of bits present in a computer, i.e., finite word length of a computer, A/D converter and memory.

Suppose it is desired to compute the matrix e^A in single precision arithmetic on computer or on a 32-bit PC. We desire roughly about 6 decimal places of precision in the fraction part of floating point numbers. Let the computation is attempted using the formula:

$$e^A = \sum_{k=0}^{oo} \frac{1}{k!} A^k \tag{1}$$

Where [A] is given as:

$$[A] = \begin{bmatrix} -49 & 24 \\ -64 & 31 \end{bmatrix} \tag{2}$$

This is easily coded and it is determined that the first 60 terms in the series suffice for the computation, in the sense that terms for k $\geq$ 60 are (10^{-7}) and no longer add anything significant to the sum. The resulting answer is:

$$e^A = \begin{bmatrix} -22.2588 & -1.43277 \\ -61.4993 & -3.47428 \end{bmatrix} \tag{3}$$

The function e^A, as defined in Equations (i) and (ii), has also, been evaluated using the MATLAB Version 7.0 simulation package [7], on an IBM PC which had Pentium 4, 2.8 Mhz, CPU. The answer obtained is

$$e^A = 1.0x10^{13} \begin{bmatrix} 0 & .0026 \\ 0 & 2.9049 \end{bmatrix} \tag{4}$$

Unfortunately, the true answer, correctly rounded is

$$e^A = \begin{bmatrix} -0.735759 & 0.551819 \\ -1.47152 & 1.10364 \end{bmatrix} \tag{5}$$

It evident that the computations involving matrices often require matrix truncation operations which lead to gross inaccurate and unacceptable results. Therefore, it becomes highly desirable to avoid such matrix truncation operations. The method presented in this paper, transformed the matrix truncation operation into a scalar truncation which can be better controlled to obtain the results of desired accuracy.

What happened here is that the intermediate term in the series got very large before the factorial began to dominate. Error is due to the finite word length. Therefore, the number of bits in word of computer must be carefully selected to achieve reliable results.

The usefulness of 32-bit microcomputers is rather limited, simply because the execution speed of 32-bit processors with single precision arithmetic is slow, yields a low throughput, and poor accuracy. The control laws of most modern aircrafts and spacecrafts are exceedingly complex and, therefore to compute a meaningful control command, the execution time and transport time delay must be minimized. A considerable reduction in execution time and improved accuracy can be achieved by employing the multi-core 32 or 64-bit microprocessors. A suitable microprocessor can be easily selected from the array of existing microprocessor.

III. Development of Proposed Method

In practice, a considerable variety of physical problems involves the transient and steady state analysis of n-dimensional coupled electric circuits. The matrix partial differential equations describing the behavior of such circuits can be written as

$$\frac{-\partial v(x,t)}{\partial x} = \left[[R] + [L] \frac{\partial}{\partial t} \right] i\ (x,t) \tag{6}$$

$$\frac{-\partial i(x,t)}{\partial x} = \left[[G] + [C] \frac{\partial}{\partial t} \right] v\ (x,t) \tag{7}$$

Taking the Laplace transform and ignoring the initial conditions, equations (1 and 2) can be written as ordinary differential equations in the compact from as

$$\frac{-dI}{dx} = [Y]\,[V] \tag{8}$$

$$\frac{-dV}{dx} = [Z]\,[I] \tag{9}$$

Where

$$[Z] = [R] + s[L] \tag{10}$$

$$[Y] = [G] + s[C] \tag{11}$$

$$[Z] = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & & & \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix} + s \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & & \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix} \tag{12}$$

Matrix [Y] is similarly defined. The [V] and [I] are n-dimensional column vectors, and are functions of parameters, the distance x, and the Laplace operator s. Form equations (3 and 4), it follows that

$$\frac{d^2 I}{dx^2} = [Y]\,[Z]\ [I] = [W_1]\,[I] \tag{13}$$

$$\frac{d^2 V}{dx^2} = [Z]\,[Y]\ [V] = [W_2][V] \tag{14}$$

The matrix product ([Z] [Y]), in general, is not equal to the matrix product ([Y] [Z]), except in special cases, when the matrices are symmetric.
The matrix equations (7 and 8) are coupled. In the case of transient, the eigenvalues and eigenvectors vary with frequency, and the modal transformation matrices must be calculated at each frequency.
However, the method proposed here does not require the computation of modal transformations, and therefore of eigenvectors. The steady state solution, with s = jw, of equations (7) and (8) leads to the different set of n-dimensional [A], [B], [C], and [D] parameter matrices of a symmetric or asymmetric n-dimensional transmission line network. The both set of generalized parameter matrices are derived.
The solution of equations (3 and 7) may be written as

$$I = \exp(x[W_1]^{1/2})\, A_1 + \exp(-x\, [W_1]^{1/2})\, A_2 \qquad (15)$$

$$V = [Y]^{-1}\, [W_1]^{1/2}\, \exp(x\, [W_1]^{1/2}) A_1 - \exp(-x[W_1]^{1/2}) A_2 \qquad (16)$$

Where A_1 and A_2 are n-dimensional column vector constants of integration, and

$$[W_1]^{1/2} = \sqrt{[Y]\,[Z]} = [a_1] + s[\beta_1] \qquad (17)$$

The eigenvalues of matrices [Z], [Y], and [W] are complex and vary with frequency. The [W] is the propagation constant matrix, and $[\alpha]$ is the attenuation constant matrix in nepers per unit length. While $[\beta]$ being a function of the displacement x and the laplace variable s is the phase constant matrix in radians per unit length, and represents a delay factor matrix.
The solution of voltage and current vectors, from equations (9 and 10) in terms of generalized ABCD coupled n-dimensional parameter matrices can be written as

$$\begin{bmatrix} [V] \\ [I] \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{bmatrix} [V_r] \\ [I_r] \end{bmatrix} \qquad (18)$$

Where

$$[A] = [Y]^{-1}\, Cosh(x[W_1]^{1/2})\, [Y] \qquad (19)$$

$$[B] = ([[Y]^{-1}\,[Z]]^{1/2})\, Sinh(x[W_1]^{1/2}) \qquad (20)$$

$$[C] = Sinh(x\, [W_1]^{1/2})([[Z]^{-1}\,[Y]]^{1/2}) \qquad (21)$$

$$[D] = Cosh(x\, [W_1]^{1/2}) \qquad (22)$$

Similarly, the solution of equations (4 and 8) yields the following set of [A], [B], [C], and [D] parameters matrices:

$$[A] = Cosh(x[W_2]^{1/2}) \qquad (23)$$

$$[B] = Sinh(x\, [W_2]^{1/2})([[Z]^{-1}\,[Y]]^{1/2}) \qquad (24)$$

$$[C] = ([[Z]^{-1}\,[Y]]^{1/2})\, Sinh(x\, [W_2]^{1/2}) \qquad (25)$$

$$[D] = [Z]^{-1}\, Cosh(x\, [W_2]^{1/2})[Z] \qquad (26)$$

Where

$$[W_2]^{1/2} = \sqrt{[Z]\,[Y]} = [a_2] + s[\beta_2] \qquad (27)$$

It may be noted that, in general, $[a_2] \neq [a_1]$ and $[\beta_2] \neq [\beta_1]$.
The [A], [B], [C], [D], $[W]^{1\backslash 2}$, and $[W]^{-1}$ are complex n-dimensional coupled non-singular square matrices.

IV. Development of T Network

The sending and receiving ends voltage and current vectors for the equivalent n-dimensional coupled T networks, given in Figure 1, can be written as

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} [U + \frac{1}{2} Z_T Y_T] & [Z_T][U + \frac{1}{4} Y_T Z_T] \\ & [U + \frac{1}{2} Y_T Z_T] \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix} \qquad (28)$$

For the n-dimensional coupled T network given in figure 1, it follows from equations (12, 17, 19, and 23) that

$$[Y_T] = [C] = [[Z]^{-1}\,[Y]]^{1/2}\, [Sinh\, (x\, [W_2]^{1/2})] \qquad (29)$$

$$\frac{1}{2}\, [Z_T] = \left[Tanh_T\, ([W_2]^{1/2}\, x/2)\right] \left[[Z]^{-1}\,[Y]\right]^{1/2} \qquad (30)$$

Where

$$\left[Tanh_T\, ([W_2]^{1/2}\, x/2)\right] = \left[Cosh[W_2]^{1/2}\, x) - [U]\right] \left[Sinh\, ([W_2]^{1/2} x)\right]^{-1} \qquad (31)$$

The equations (13 and 15) could be equally used.

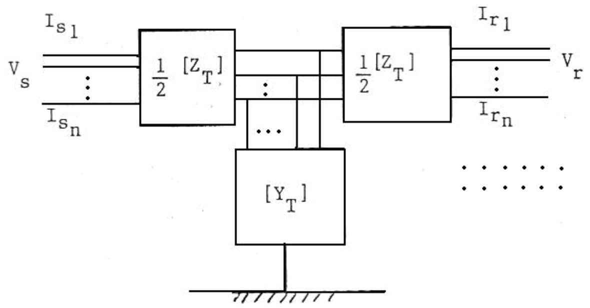

Figure 1: N-Dimensional Coupled T Network

V. DEVELOPMENT OF PI NETWORK

The sending and receiving ends voltage and current vectors for the n-dimensional coupled PI networks, shown in Figure 2, Can be written as

$$\begin{bmatrix} V_s \\ I_s \end{bmatrix} = \begin{bmatrix} [U + \frac{1}{2} Z_\pi Y_\pi] & [Z_\pi] \\ [Y_\pi][U + \frac{1}{2} Z_\pi Y_\pi] & [U + \frac{1}{2} Y_\pi Z_\pi] \end{bmatrix} \begin{bmatrix} V_r \\ I_r \end{bmatrix} \tag{32}$$

For the n-dimensional coupled Pi network given in Figure 2, it follows from equations (12, 17, 18, and 26) that

$$[Z_\pi] = [Sinh\ (x[W_2]^{1/2})\ [[Z]^{-1}[Y]]^{1/2} \tag{33}$$

$$\frac{1}{2}[Y_\pi] = [[Z]^{-1}[Y]]^{1/2}\ Tanh_\pi([W_2]^{1/2}\ x/2)] \tag{34}$$

Where

$$\left[Tanh_\pi\ ([W_2]^{1/2}\ x/2)\right] = \left[Sinh\ [W_2]^{1/2}\ x)^{-1}\right]\left[Cosh([W_2]^{1/2}\ x) - U\right] \tag{35}$$

The equations (13 and 14) could be equally used.

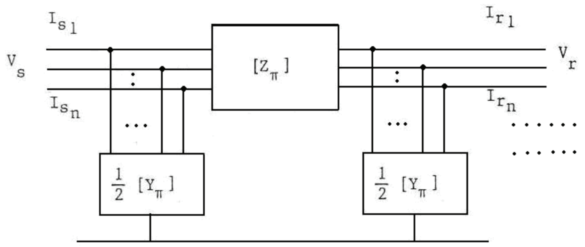

Figure 2: N-Dimensional Coupled Pi Network

VI. PROPOSED METHOD OF SOLUTION

The [A], [B], [C], [D], [Z_T], and [Y_π] are complex n-dimensional coupled non-singular square matrices. These can be computed using the Cayley-Hamilton's theorem, and thereby avoiding the evaluation of infinite series representing the hyperbolic functions having n-dimensional coupled matrices as their arguments. Using Cayley-Hamilton's theorem, the matrix [A] in Equation (17), for example, can be computed for a particular frequency as follows [5]:

$$[A] = Cosh(x[W_2]^{1/2}) = Cosh\ [P] \tag{36}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of [P]

$$\begin{bmatrix} 1\ \lambda_1 & \lambda_1^2 \ldots \lambda_1^{n-1} \\ 1\ \lambda_2 & \lambda_2^2 \ldots \lambda_2^{n-1} \\ \cdots & \cdots \\ \cdots & \cdots \\ 1\ \lambda_n & \lambda_n^2 \ldots \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdots \\ \cdots \\ \alpha_{n-1} \end{bmatrix} = \begin{bmatrix} Cosh\ \lambda_1 \\ Cosh\ \lambda_2 \\ \cdots \\ \cdots \\ Cosh\ \lambda_n \end{bmatrix} \tag{37}$$

$\alpha_0, \alpha_1, \alpha_1 \cdots, \alpha_{n-1}$ are the Cayley-Hamilton's constants. Using equations (30 and 31), it follows from Cayley-Hamilton's theorem that

$$[A] = Cosh[P] = a_0 I + a_1 P + a_2 P^2 + \ldots + a_{n-1} P^{n-1} \tag{38}$$

Where

$$[A] = \begin{bmatrix} A_{11}\ A_{12}\ \ldots\ A_{1n} \\ A_{21}\ A_{22}\ \ldots\ A_{2n} \\ \cdots \\ \cdots \\ A_{n1}\ A_{n2}\ \ldots\ A_{nn} \end{bmatrix} \tag{39}$$

VII. COMPUTATION OF SQUARE-ROOT OF [W_2] MATRIX

The square root a complex matrix [W_2] can be computed using the Cayley-Hamilton's theorm as follows [5]:

$$[W_2]^{1/2} = [P] \tag{40}$$

Let $\beta_1\ \beta_2 \ldots, \beta_n$ be the eigenvalue of [W_2].

$$\begin{bmatrix} 1\ \beta_1\ \beta_1^2 \ldots \beta_1^{n-1} \\ 1\ \beta_2\ \beta_2^2 \ldots \beta_2^{n-1} \\ \cdots \\ \cdots \\ 1\ \beta_n\ \beta_n^2 \ldots \beta_n^{n-1} \end{bmatrix} \begin{bmatrix} k_o \\ k_1 \\ \vdots \\ \vdots \\ k_{n-1} \end{bmatrix} = \begin{bmatrix} \beta_1^{1/2} \\ \beta_2^{1/2} \\ \vdots \\ \vdots \\ \beta_n^{1/2} \end{bmatrix} \tag{41}$$

$K_o, k_1, \ldots, k_{n-1}$, are the Cayley-Hamilton's constants. Using equations (34) and (35), it follows from Cayley-Hamilton's theorem that

$$[P] = [W_2]^{1/2} = k_o I + k_1 W_2 + k_2 W_2^2 + \ldots + k_{n-1} W_2^{n-1} \tag{42}$$

Where

$$[A] = \begin{bmatrix} P_{11}\ P_{12}\ \ldots\ P_{1n} \\ P_{21}\ P_{22}\ \ldots\ P_{2n} \\ \cdots \\ \cdots \\ P_{n1}\ P_{n2}\ \ldots\ P_{nn} \end{bmatrix} \tag{43}$$

It may be noted that the square root of the complex matrix [W_2] will also be computed by using Cayley-Hamilton's theorem [5]. The remaining [Z_T], [Y_π], and [B], [C], and [D] generalized constants' parameter matrices can be similarly computed. System matrices having repeated eigenvalues will require differentiation and can be easily handled by this method. In such cases, the use of eigenvector based modal transformations may produce Jordan system matrices which having coupled Jordan elements will be difficult to solve with conventional methods.

CONCLUSION

A new computer based method for transient and steady state solution of n-dimensional coupled transmission line networks or communication circuits with distributed parameters has been presented. In this method, a novel approach has been developed for formulating and computing the n-dimensional generalized ABCD parameter matrices, and for developing and solving the equivalent n-dimensional coupled T and Pi networks with distributed parameters.

The method of solution does not require the computation of modal transformations, and therefore, of eigenvectors. Proposed solution utilizes the Cayley-Hamilton's theorem, and does not involve the evaluation of infinite series of hyperbolic functions having n-dimensional coupled matrices as their arguments. In this method the generalized ABCD parameter matrices which are fundamental to the solution and development of such equivalent n-dimensional T and Pi networks are computed with finite terms. The square root function of the complex [W_2] matrix is computed by this method with finite terms. As a result, truncation of matrices is eliminated, and an improved closed from solution is obtained. The method is straight forward, computationally efficient, and neither it involves the use of eigenvector based modal transformations necessary for diagonalization of parameter matrices nor does it require the evaluation of infinite series of hyperbolic functions with n-dimensional matrices as their arguments, and is extremely useful in the steady state and transient analysis of n-dimensional, unbalanced, coupled systems with distributed parameters.

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