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ORIGINAL ARTICLE

Analytic solutions of Oldroyd-B fluid with fractional derivatives in a circular duct that applies a constant couple

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Abstract The aim of this article was to analyze the rotational flow of an Oldroyd-B fluid with fractional derivatives, induced by an infinite circular cylinder that applies a constant couple to the fluid. Such kind of problem in the settings of fractional derivatives has not been found in the literature. The solutions are based on an important remark regarding the governing equation for the non-trivial shear stress. The solutions that have been obtained satisfy all imposed initial and boundary conditions and can easily be reduced to the similar solutions corresponding to ordinary Oldroyd-B, fractional/ordinary Maxwell, fractional/ordinary second-grade, and Newtonian fluids performing the same motion. The obtained results are expressed in terms of Newtonian and non-Newtonian contributions. Finally, the influence of fractional parameters on the velocity, shear stress and a comparison between generalized and ordinary fluids is graphically underlined.

1. Introduction

Non-Newtonian fluids play a paramount role in a large number of industries and the branches of knowledge concerned with the applied sciences. Among the numerous models that have been used to portray the behavior of non-Newtonian fluids, the Oldroyd-B model seems to be agreeable to examine and more prominently experiment. This model can describe stress-relaxation, creep and usual stress differences is the most successful model for relating the response of many dilute polymeric liquids. It contains, as special cases the Maxwell and viscous fluid models. The motion of a fluid in the neighborhood of a moving body is of great interest for industry. The flow between cylinders or through a rotating cylinder has applications in the food industry and being one of the most important and interesting problems of motion near rotating bodies. The velocity distribution for different motions of Newtonian fluids through a circular cylinder is given in [1]. Accurate solutions regarding motions of non-Newtonian fluids in cylindrical domains appear to be those of Ting [2], Srivastava [3] and Waters and King [4] for second grade, Maxwell and Oldroyd-B fluids respectively. Through the time numerous

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papers concerning such motions of non-Newtonian fluids have been published. Among them, we here recall only a small number of those regarding Oldroyd-B fluids [5–19].

Though, all above-mentioned papers incorporate motion problems in which the velocity is given on the boundary. Although in some realistic problems what is specified is the shear stress [20,21], more precisely the force with which the cylinder is moved. To restate, in Newtonian mechanics force is the reason and kinematics is the effect (see Rajagopal [23] for a comprehensive discussion on the same). Therefore, the boundary condition on stresses is significant, and Renardy [22] illustrated how well-posed boundary value problems can be formulated in this way. In the last time numerous solutions for such motions of rate type fluids have been recognized (see [24–29] and the references therein), but all these problems respond to differential expressions of the stress on the boundary. This is due to their governing equations that, unlike those corresponding to Newtonian and second grade fluids, contain differential expressions acting on the non-trivial shear stresses.

With these motivations, the aim of this article was to analyze the rotational flow of an Oldroyd-B fluid with fractional derivatives induced by an infinite circular cylinder that applies a constant couple to the fluid. These solutions are obtained by means of integral transforms. The obtained solutions satisfy all imposed initial and boundary conditions and expressed in terms of Newtonian and non-Newtonian contributions that can be reduced to the similar solutions corresponding to ordinary Oldroyd-B [30], fractional Maxwell, ordinary Maxwell, fractional/ordinary second-grade and Newtonian fluids.

2. Mathematical formulation of the problem

The Cauchy-stress tensor $\mathbf{T}$ corresponding to an incompressible Oldroyd-B fluid is related to the fluid motion by the relations

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu[\mathbf{A} + \lambda_r(\dot{\mathbf{A}} - \mathbf{LA} - \mathbf{AL}^T)], \tag{1}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor, μ is the dynamic viscosity, λ and λ_r are the relaxation and retardation times respectively. The superscript T indicates the transpose operation and the superposed dot denotes the material time derivative. The model characterized by the constitutive Eq. (1) contains as special cases the upper convected Maxwell model for $\lambda_r = 0$ and the Newtonian fluid model for $\lambda = \lambda_r = 0$. In some special cases like that to be here considered, the governing equations for Oldroyd-B fluids resemble to those for second-grade fluids.

Let us assume that an incompressible Oldroyd-B fluid is at rest in an infinite circular cylinder of radius R. At time $t = 0^+$, the cylinder begins to turn about its axis due to constant torque per unit length $2\pi Rf$. Owing to the shear, the fluid is gradually moved and we are looking for a velocity field of the form

$$\mathbf{V} = \omega(r, \theta)\mathbf{e}_\theta, \tag{2}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ-direction of the system of Cylindrical coordinates.

For such a flow, the constraint of incompressibility is identically satisfied. We also assume that the extra-stress tensor $\mathbf{S}$, as well as the velocity $\mathbf{V}$ is a function of time and radius only. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(r, 0) = \mathbf{0}, \quad \mathbf{S}(r, 0) = \mathbf{0}, \tag{3}$$

and the constitutive Eq. (1) implies $S_{rr} = S_{rz} = S_{\theta z} = S_{zz} = 0$ and the meaningful partial differential equation

$$\left(1 + \lambda\frac{\partial}{\partial t}\right)\tau(r, t) = \mu\left(1 + \lambda_r\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r, t), \tag{4}$$

where $\tau(r, t) = S_{r\theta}(r, t)$ is the non-trivial stress. Neglecting body forces, the balance of linear momentum leads to the relevant equation due to rotation symmetry

$$\rho\frac{\partial\omega(r, t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\tau(r, t), \tag{5}$$

where ρ is the constant density of the fluid.

Usually in the literature, the governing equation for the velocity is obtained by eliminating $\tau(r, t)$ between Eqs. (4) and (5). Since our interest here is to solve a motion problem with shear stress on the boundary, we follow the pattern and eliminate $\omega(r, t)$ in order to get the governing equation

$$\left(1 + \lambda^\alpha\frac{\partial}{\partial t}\right)\frac{\partial\tau(r, t)}{\partial t} = \nu\left(1 + \lambda_r^\beta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tau(r, t), \tag{6}$$

for the shear stress $\tau(r, t)$. Here $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the fluid.

Its corresponding fractional model is

$$(1 + \lambda^\alpha D_t^\alpha)\frac{\partial\tau(r, t)}{\partial t} = \nu(1 + \lambda_r^\beta D_t^\beta)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\tau(r, t), \tag{7}$$

where

$$\begin{cases} D_t^\alpha(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, & m-1 < \alpha < m, \ m \in \mathbb{N} \\ D_t^m f(t) = \frac{d^m f(t)}{dt^m}, & m = 1, 2, \ldots \end{cases} \tag{8}$$

is the Caputo fractional derivative operator [31–33].

The appropriate initial and boundary conditions are:

$$\tau(r, 0) = \left.\frac{\partial\tau(r, t)}{\partial t}\right|_{t=0} = 0, \qquad r \in [0, R], \tag{9}$$

$$\tau(R, t) = fH(t), \qquad t > 0. \tag{10}$$

3. Solution of the problem

Applying the Laplace transform to Eq. (7) and keeping in mind the initial conditions Eq. (9), we have

$$\bar{\tau}(r, q) = \frac{\nu(1 + \lambda_r^\beta q^\beta)}{q + \lambda^\alpha q^{\alpha+1}}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right)\bar{\tau}(r, q), \tag{11}$$

and the equation Eq. (10) becomes

$$\bar{\tau}(R, q) = \frac{f}{q},$$

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where $\overline{\tau}(R,q)$ and $\overline{\tau}(r,q)$ are the Laplace transform of the function $\tau(R,t)$ and $\tau(r,t)$ respectively. Multiplying Eq. (11) by $rJ_2(rr_n)$ where r_n are the positive roots of $J_2(Rr)=0$, integrating the result from 0 to R and use the identities

$$\int_0^R rJ_2(rr_n)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\overline{\tau}(r,q)dr = -Rr_nJ_2'(Rr_n)\overline{\tau}(R,q)-r_n^2\overline{\tau}(r_n,q) \tag{12}$$

$$\int_0^R rJ_2(rr_n)\overline{\tau}(r,q)dr = \frac{\nu(1+\lambda_r^\beta q^\beta)}{q+\lambda^\alpha q^{\alpha+1}} \times\int_0^R rJ_2(rr_n)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2}\right)\overline{\tau}(r,q)dr. \tag{13}$$

Taking Hankel transform [34, Section 14, Eq. (59)], and using Eq. (12)

$$\overline{\tau}_H(r_n,q) = -\frac{\nu fRr_n(1+\lambda_r^\beta q^\beta)J_2'(Rr_n)}{q[q+\lambda^\alpha q^{\alpha+1}+\nu(1+\lambda_r^\beta q^\beta)r_n^2]}. \tag{14}$$

In order to inverting Eq. (14), it can be written in suitable form

$$\overline{\tau}_H(r_n,q)=\frac{1}{\nu r_n^2}\left[\frac{1}{q}-\frac{1}{q+\nu r_n^2}\right]-\lambda_r^\beta\frac{q^{\beta-1}}{q+\nu r_n^2}+\left[\frac{q^{\beta-1}}{q+\nu r_n^2}+\lambda_r^\beta\frac{q^{2\beta-1}}{q+\nu r_n^2}\right]$$
$$\left[\sum_{p=0}^\infty\sum_{j=0}^\infty\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\lambda_r^{\beta j}\times\left(\frac{q^{\alpha-\beta+\beta j-p}}{(q^\alpha+\lambda^{-\alpha})^{p+1}}+a\nu r_n^2\frac{q^{\beta j-p-1}}{(q^\alpha+\lambda^{-\alpha})^{p+1}}\right)\right]. \tag{15}$$

Applying the inverse Laplace and the Hankel transforms and use the identity [34, the entry 1 of Table X],

$$r^2=-2R\sum_{n=1}^\infty\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}.$$

Lengthy but straightforward computations show that

$$\tau(r,t)=\frac{r^2fH(t)}{R^2}-\frac{2f}{R}\sum_{n=1}^\infty\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}e^{-\nu r_n^2t}$$
$$-\frac{2f\nu}{R}\sum_{n=1}^\infty r_n\frac{J_2(rr_n)}{J_1(Rr_n)}\lambda_r^\beta G_{1,\beta-1,1}(-\nu r_n^2,t)$$
$$+\frac{2f\nu}{R}\sum_{n=1}^\infty r_n\frac{J_2(rr_n)}{J_1(Rr_n)}\int_0^t[G_{1,\beta-1,1}(-\nu r_n^2,\tau-s)$$
$$+\lambda_r^\beta G_{1,2\beta-1,1}(-\nu r_n^2,\tau-s)]$$
$$\times\left[\sum_{p=0}^\infty\sum_{j=0}^\infty\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\lambda_r^{\beta j}\left(G_{\alpha,\alpha-\beta+j\beta-p,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right.\right.$$
$$\left.\left.+a\nu r_n^2G_{\alpha,j\beta-\beta-1,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right)\right]ds. \tag{16}$$

The fractional model of Eq. (5),

$$\rho D_t^\alpha\omega(r,t)=\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\tau(r,t). \tag{17}$$

By introducing Eq. (16) into Eq. (17), we obtain

$$\rho D_t^\alpha\omega(r,t)=\frac{2rfH(t)}{R^2}-\frac{2f}{R}\sum_{n=1}^\infty\frac{\frac{\partial J_2(rr_n)}{\partial r}}{r_nJ_1(Rr_n)}e^{-\nu r_n^2t}$$
$$-\frac{2f\nu}{R}\sum_{n=1}^\infty r_n\frac{\frac{\partial J_2(rr_n)}{\partial r}}{J_1(Rr_n)}\lambda_r^\beta G_{1,\beta-1,1}(-\nu r_n^2,t)$$
$$+\frac{2f\nu}{R}\sum_{n=1}^\infty r_n\frac{\frac{\partial J_2(rr_n)}{\partial r}}{J_1(Rr_n)}\int_0^t[G_{1,\beta-1,1}(-\nu r_n^2,\tau-s)$$
$$+\lambda_r^\beta G_{1,2\beta-1,1}(-\nu r_n^2,\tau-s)]$$
$$\times\left[\sum_{p=0}^\infty\sum_{j=0}^\infty\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\lambda_r^{\beta j}\left(G_{\alpha,\alpha-\beta+j\beta-p,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right.\right.$$
$$\left.\left.+a\nu r_n^2G_{\alpha,j\beta-\beta-1,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right)\right]ds$$
$$+\frac{2rfH(t)}{R^2}-\frac{4f}{rR}\sum_{n=1}^\infty\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}e^{-\nu r_n^2t}$$
$$-\frac{4f\nu}{rR}\sum_{n=1}^\infty r_n\frac{J_2(rr_n)}{J_1(Rr_n)}\lambda_r^\beta G_{1,\beta-1,1}(-\nu r_n^2,t)$$
$$+\frac{4f\nu}{rR}\sum_{n=1}^\infty r_n\frac{J_2(rr_n)}{J_1(Rr_n)}\int_0^t[G_{1,\beta-1,1}(-\nu r_n^2,\tau-s)$$
$$+\lambda_r^\beta G_{1,2\beta-1,1}(-\nu r_n^2,\tau-s)]$$
$$\times\left[\sum_{p=0}^\infty\sum_{j=0}^\infty\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\lambda_r^{\beta j}\left(G_{\alpha,\alpha-\beta+j\beta-p,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right.\right.$$
$$\left.\left.+a\nu r_n^2G_{\alpha,j\beta-\beta-1,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right)\right]ds, \tag{18}$$

or

$$D_t^\alpha\omega(r,t)=\frac{4rfH(t)}{\rho R^2}-\frac{2f}{\rho R}\sum_{n=1}^\infty\frac{J_1(rr_n)}{r_nJ_1(Rr_n)}e^{-\nu r_n^2t}$$
$$-\frac{2f\nu}{\rho R}\sum_{n=1}^\infty r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\lambda_r^\beta G_{1,\beta-1,1}(-\nu r_n^2,t)$$
$$+\frac{2f\nu}{\rho R}\sum_{n=1}^\infty r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\int_0^t[G_{1,\beta-1,1}(-\nu r_n^2,\tau-s)$$
$$+\lambda_r^\beta G_{1,2\beta-1,1}(-\nu r_n^2,\tau-s)]$$
$$\times\left[\sum_{p=0}^\infty\sum_{j=0}^\infty\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\lambda_r^{\beta j}\left(G_{\alpha,\alpha-\beta+j\beta-p,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right.\right.$$
$$\left.\left.+a\nu r_n^2G_{\alpha,j\beta-\beta-1,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right)\right]ds. \tag{19}$$

Applying Laplace transform to Eq. (19) and use the initial condition Eq. $(3)_1$, we get

$$\overline{\omega}(r,q)=\frac{4rfH(t)}{\rho R^2}\frac{1}{q^{\alpha+1}}-\frac{2f}{\rho R}\sum_{n=1}^\infty\frac{J_1(rr_n)}{r_nJ_1(Rr_n)}\frac{q^{-\alpha}}{q+\nu r_n^2}$$
$$-\frac{2f\nu}{\rho R}\sum_{n=1}^\infty r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\lambda_r^\beta\frac{q^{\beta-\alpha-1}}{q+\nu r_n^2}$$
$$+\frac{2f\nu}{\rho R}\sum_{n=1}^\infty r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\left[\frac{q^{\beta-\alpha-1}}{q+\nu r_n^2}+\lambda_r^\beta\frac{q^{2\beta-\alpha-1}}{q+\nu r_n^2}\right]$$
$$\times\left[\sum_{p=0}^\infty\sum_{j=0}^\infty\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\lambda_r^{\beta j}\right.$$
$$\left.\times\left(\frac{q^{\alpha-\beta+\beta j-p}}{(q^\alpha+\lambda^{-\alpha})^{p+1}}+a\nu r_n^2\frac{q^{\beta j-p-1}}{(q^\alpha+\lambda^{-\alpha})^{p+1}}\right)\right]. \tag{20}$$

Applying inverse Laplace transform to Eq. (20), we have

$$\begin{aligned}\omega(r,t) &= \frac{4rf}{\rho R^2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2f}{\rho R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_1(Rr_n)}G_{1,-\alpha,1}(-\nu r_n^2,t)\\ &-\frac{2f\nu}{\rho R}\sum_{n=1}^{\infty}r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\lambda_r^\beta G_{1,\beta-\alpha-1,1}(-\nu r_n^2,t)\\ &+\frac{2f\nu}{\rho R}\sum_{n=1}^{\infty}r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\int_0^t\big[G_{1,\beta-\alpha-1,1}(-\nu r_n^2,\tau-s)\\ &+\lambda_r^\beta G_{1,2\beta-\alpha-1,1}(-\nu r_n^2,\tau-s)\big]\\ &\times\left[\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\lambda_r^{\beta j}\left(G_{\alpha,\alpha-\beta+j\beta-p,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right.\right.\\ &\left.\left.+a\nu r_n^2G_{\alpha,j\beta-\beta-1,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right)\right]ds. \qquad (21)\end{aligned}$$

4. Limiting cases

4.1. Case $\alpha=\beta\to\mathbf{1}$, solutions corresponding to ordinary Oldroyd-B fluid

By letting $\alpha=\beta\to1$ into Eqs. (16) and (21), we obtain

$$\begin{aligned}\tau_{OB}(r,t) &= \frac{r^2fH(t)}{R^2}-\frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}e^{-\nu r_n^2t}\\ &-\frac{2f\nu}{R}\sum_{n=1}^{\infty}r_n\frac{J_2(rr_n)}{J_1(Rr_n)}\lambda_rG_{1,0,1}(-\nu r_n^2,t)\\ &+\frac{2f\nu}{R}\sum_{n=1}^{\infty}r_n\frac{J_2(rr_n)}{J_1(Rr_n)}\int_0^t\big[G_{1,0,1}(-\nu r_n^2,\tau-s)\\ &+\lambda_r^\beta G_{1,1,1}(-\nu r_n^2,\tau-s)\big]\\ &\times\left[\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda}\right)^p\lambda_r^j\left(G_{1,j-p,p+1}\left(-\frac{1}{\lambda},s\right)\right.\right.\\ &\left.\left.+\frac{\lambda_r}{\lambda}\nu r_n^2G_{1,j,p+1}\left(-\frac{1}{\lambda},s\right)\right)\right]ds, \qquad (22)\end{aligned}$$

$$\begin{aligned}\omega_{OB}(r,t) &= \frac{4rft}{\rho R^2}-\frac{2f}{\rho R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_1(Rr_n)}G_{1,-1,1}(-\nu r_n^2,t)\\ &-\frac{2f\nu}{\rho R}\sum_{n=1}^{\infty}r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\lambda_rG_{1,-1,1}(-\nu r_n^2,t)\\ &+\frac{2f\nu}{\rho R}\sum_{n=1}^{\infty}r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\int_0^t\big[G_{1,-1,1}(-\nu r_n^2,\tau-s)\\ &+\lambda_rG_{1,0,1}(-\nu r_n^2,\tau-s)\big]\\ &\times\left[\sum_{p=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^p p!}{(p-j)!j!}\left(\frac{\nu r_n^2}{\lambda}\right)^p\lambda_r^j\left(G_{1,j-p,p+1}\left(-\frac{1}{\lambda},s\right)\right.\right.\\ &\left.\left.+\frac{\lambda_r}{\lambda}\nu r_n^2G_{1,j,p+1}\left(-\frac{1}{\lambda},s\right)\right)\right]ds. \qquad (23)\end{aligned}$$

As a check of our results we plot the graphs for Eqs. (22) and (23) to justify [30, Eqs. (15) and (17)] (see Figs. 1 and 2).

4.2. Case $\lambda_r^\beta\to\mathbf{0},\ \beta\to\mathbf{0}$, solutions corresponding to generalized Maxwell fluid

By now letting $\lambda_r^\beta\to0,\ \beta\to0$ into Eqs. (16) and (21), we have

$$\begin{aligned}\tau_{GM}(r,t) &= \frac{r^2fH(t)}{R^2}-\frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_nJ_1(Rr_n)}e^{-\nu r_n^2t}\\ &+\frac{2f\nu}{R}\sum_{n=1}^{\infty}r_n\frac{J_2(rr_n)}{J_1(Rr_n)}\\ &\times\int_0^t\big[G_{1,-1,1}(-\nu r_n^2,\tau-s)\big]\\ &\times\left[\sum_{p=0}^{\infty}(-1)^p\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\left(G_{1,-p,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right)\right]ds, \qquad (24)\end{aligned}$$

$$\begin{aligned}\omega_{GM}(r,t) &= \frac{4rf}{\rho R^2}\frac{t^\alpha}{\Gamma(\alpha+1)}\\ &-\frac{2f}{\rho R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_nJ_1(Rr_n)}\left[G_{1,-\alpha,1}(-\nu r_n^2,t)-\frac{t^\alpha}{\Gamma(\alpha+1)}\right]\\ &+\frac{2f\nu}{\rho R}\sum_{n=1}^{\infty}r_n\frac{J_1(rr_n)}{J_1(Rr_n)}\times\int_0^t\big[G_{1,-\alpha-1,1}(-\nu r_n^2,\tau-s)\big]\\ &\times\left[\sum_{p=0}^{\infty}(-1)^p\left(\frac{\nu r_n^2}{\lambda^\alpha}\right)^p\left(G_{\alpha,\alpha-p,p+1}\left(-\frac{1}{\lambda^\alpha},s\right)\right]ds. \qquad (25)\end{aligned}$$

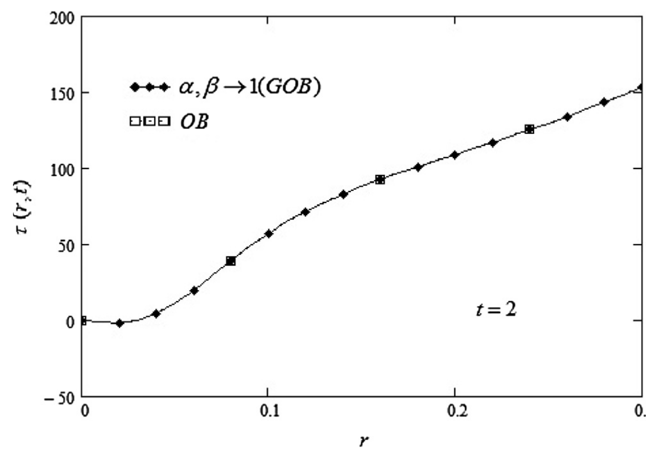

Figure 1 Comparison of shear stress $\tau(r,t)$ versus r with $\lambda_r=0.15,\ \lambda=0.85,\ t=2$ sec of Eq. (22) and [30, Eq. (15)].

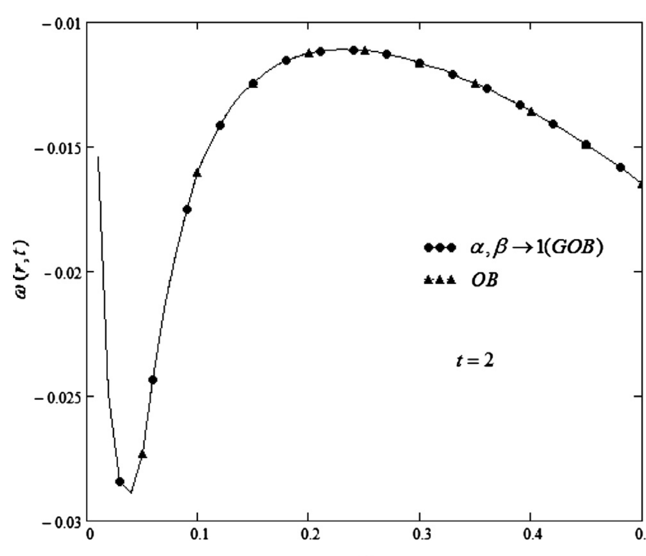

Figure 2 Comparison of velocity $\omega(r,t)$ versus r with $\lambda_r=0.15$, $\lambda=0.85,\ t=2$ sec of Eq. (22) and [30, Eq. (17)].

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4.3. Case $\lambda_r \to 0$ or $\alpha \to 1$, solutions corresponding ordinary Maxwell fluid

We can find shear stress and velocity profile of Maxwell fluid by using $\lambda_r \to 0$ into Eqs. (22) and (23), or by letting $\alpha \to 1$ into Eqs. (24) and (25), we have

$$\tau_M(r,t) = \frac{r^2 f H(t)}{R^2} - \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n J_1(Rr_n)}e^{-\nu r_n^2 t} + \frac{2f\nu}{R}\sum_{n=1}^{\infty} r_n \frac{J_2(rr_n)}{J_1(Rr_n)} \times \int_0^t \left[G_{1,0,1}(-\nu r_n^2, \tau - s)\right] \times \left[\sum_{p=0}^{\infty}(-1)^p\left(\frac{\nu r_n^2}{\lambda}\right)^p\left(G_{1,-p,p+1}\left(-\frac{1}{\lambda}, s\right)\right)\right] ds, \quad (26)$$

$$\omega_M(r,t) = \frac{4rft}{\rho R^2} - \frac{2f}{\rho R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_1(Rr_n)}G_{1,-1,1}(-\nu r_n^2, t) - t + \frac{2f\nu}{\rho R}\sum_{n=1}^{\infty} r_n \frac{J_1(rr_n)}{J_1(Rr_n)} \times \int_0^t \left[G_{1,-1,1}(-\nu r_n^2, \tau - s)\right] \times \left[\sum_{p=0}^{\infty}(-1)^p\left(\frac{\nu r_n^2}{\lambda}\right)^p\left(G_{1,j-p,p+1}\left(-\frac{1}{\lambda}, s\right)\right)\right] ds. \quad (27)$$

4.4. Case $\lambda^\alpha \to 0$, $\alpha \to 0$, solutions corresponding generalized second grade fluid

Putting limit $\lambda^\alpha \to 0$ into Eqs. (16) and (21), we have shear stress and velocity profile of generalized second grade fluid

$$\tau_{GSG}(r,t) = \frac{r^2 f H(t)}{R^2} - \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n J_1(Rr_n)}e^{-\nu r_n^2 t} - \frac{2f\nu}{R}\sum_{n=1}^{\infty} r_n \frac{J_2(rr_n)}{J_1(Rr_n)}\lambda_r^\beta G_{1,\beta-1,1}(-\nu r_n^2, t) + \frac{2f\nu}{R}\sum_{n=1}^{\infty} r_n \frac{J_2(rr_n)}{J_1(Rr_n)}\int_0^t \times \left[G_{1,\beta-1,1}(-\nu r_n^2, \tau - s) + \lambda_r^\beta G_{1,2\beta-1,1}(-\nu r_n^2, \tau - s)\right] \times \left[\sum_{p=0}^{\infty}(-1)^p\left(\nu r_n^2 \lambda_r^\beta\right)^p\left(G_{1,p\beta,p+1}(-\nu r_n^2, s)\right)\right] ds, \quad (28)$$

$$\omega_{GSG}(r,t) = \frac{4rf}{\rho R^2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2f}{\rho R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_1(Rr_n)}G_{1,-\alpha,1}(-\nu r_n^2, t) - \frac{2f\nu}{\rho R}\sum_{n=1}^{\infty} r_n \frac{J_1(rr_n)}{J_1(Rr_n)}\lambda_r^\beta G_{1,\beta-1,1}(-\nu r_n^2, t) + \frac{2f\nu}{\rho R}\sum_{n=1}^{\infty} r_n \frac{J_1(rr_n)}{J_1(Rr_n)} \times \int_0^t \left[G_{1,\beta-1,1}(-\nu r_n^2, \tau - s) + \lambda_r^\beta G_{1,2\beta-1,1}(-\nu r_n^2, \tau - s)\right] \times \left[\sum_{p=0}^{\infty}(-1)^p\left(\nu r_n^2 \lambda_r^\beta\right)^p\left(G_{1,p\beta,p+1}(-\nu r_n^2, s)\right)\right] ds. \quad (29)$$

Furthermore, when $\beta \to 1$ into Eqs. (28) and (29), we have shear stress and velocity fields for ordinary second grade fluid

$$\tau_{SG}(r,t) = \frac{r^2 f H(t)}{R^2} - \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n J_1(Rr_n)}e^{-\nu r_n^2 t} - \frac{2f\nu}{R}\sum_{n=1}^{\infty} r_n \times \frac{J_2(rr_n)}{J_1(Rr_n)}\lambda_r G_{1,0,1}(-\nu r_n^2, t) + \frac{2f\nu}{R}\sum_{n=1}^{\infty} r_n \frac{J_2(rr_n)}{J_1(Rr_n)} \times \int_0^t \left[G_{1,0,1}(-\nu r_n^2, \tau - s) + \lambda_r^\beta G_{1,1,1}(-\nu r_n^2, \tau - s)\right] \times \left[\sum_{p=0}^{\infty}(-1)^p\left(\nu r_n^2 \lambda_r\right)^p\left(G_{1,p,p+1}(-\nu r_n^2, s)\right)\right] ds, \quad (30)$$

$$\omega_{SG}(r,t) = \frac{4rf}{\rho R^2}\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{2f}{\rho R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_1(Rr_n)}G_{1,-\alpha,1}(-\nu r_n^2, t) - \frac{2f\nu}{\rho R}\sum_{n=1}^{\infty} r_n \frac{J_1(rr_n)}{J_1(Rr_n)}\lambda_r G_{1,0,1}(-\nu r_n^2, t) + \frac{2f\nu}{\rho R}\sum_{n=1}^{\infty} r_n \frac{J_1(rr_n)}{J_1(Rr_n)} \times \int_0^t \left[G_{1,0,1}(-\nu r_n^2, \tau - s) + \lambda_r G_{1,1,1}(-\nu r_n^2, \tau - s)\right] \times \left[\sum_{p=0}^{\infty}(-1)^p\left(\nu r_n^2 \lambda_r\right)^p\left(G_{1,p,p+1}(-\nu r_n^2, s)\right)\right] ds. \quad (31)$$

4.5. Case $\lambda^\alpha \to 0$ and $\lambda_r^\beta \to 0$ solutions corresponding to Newtonian fluid

Finally by making $\lambda^\alpha \to 0$ and $\lambda_r^\beta \to 0$ into Eqs. (16) and (21)

$$\tau_N(r,t) = \frac{r^2 f H(t)}{R^2} + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)}{r_n J_1(Rr_n)}e^{-\nu r_n^2 t}, \quad (32)$$

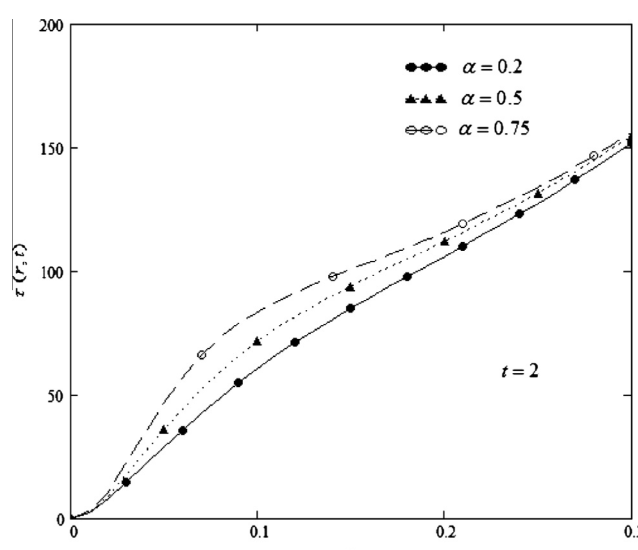

Figure 3 Shear stress $\tau(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\beta = 0.8$, $t = 2$ sec and different values of α.

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$$\omega_N(r,t) = \frac{4rft}{\rho R^2} + \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^2 J_1(Rr_n)}\left[1 - e^{-\nu r_n^2 t}\right], \tag{33}$$

we recovered the solutions which are identical to those obtained in [30, Eqs. (22) and (23)].

5. Conclusion

This study highlights the key features of the rotational flow of an Oldroyd-B fluid with fractional derivatives induced by an infinite circular cylinder that applies a constant couple to the fluid. Such kind of problem has not been solved in the literature. The present solutions are based on a simple but important remark regarding the governing equation for the

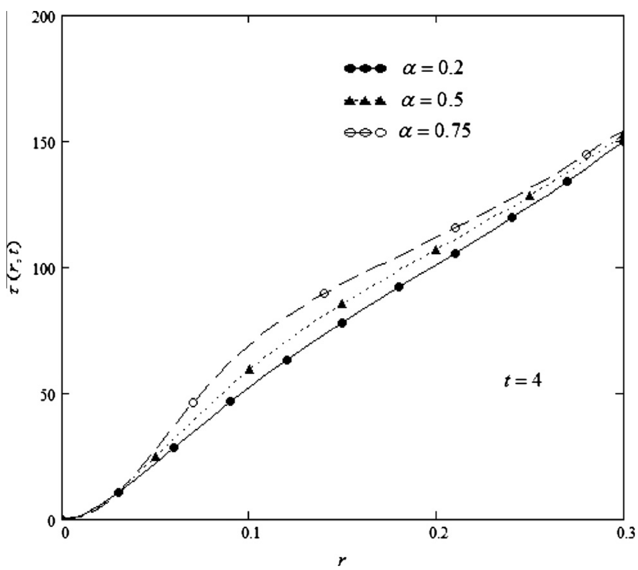

Figure 4 Shear stress $\tau(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\beta = 0.8$, $t = 4$ sec and different values of α.

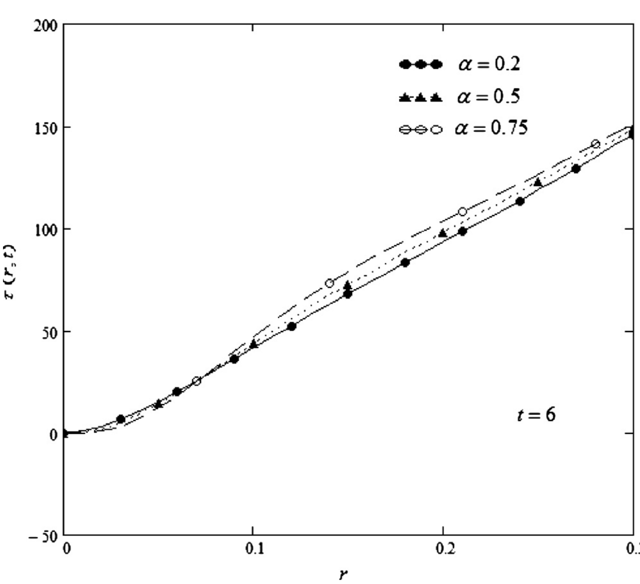

Figure 5 Shear stress $\tau(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\beta = 0.8$, $t = 6$ sec and different values of α.

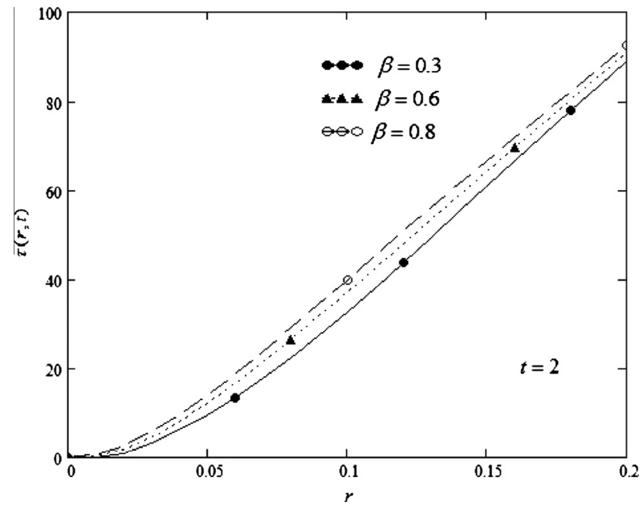

Figure 6 Shear stress $\tau(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\alpha = 0.2$, $t = 2$ sec and different values of β.

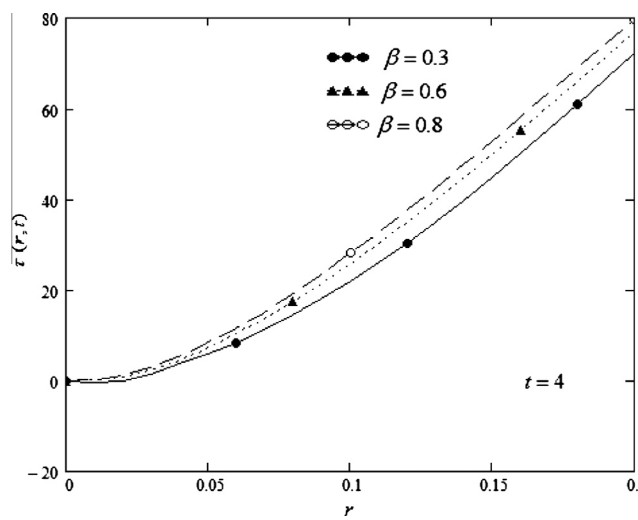

Figure 7 Shear stress $\tau(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\alpha = 0.2$, $t = 4$ sec and different values of β.

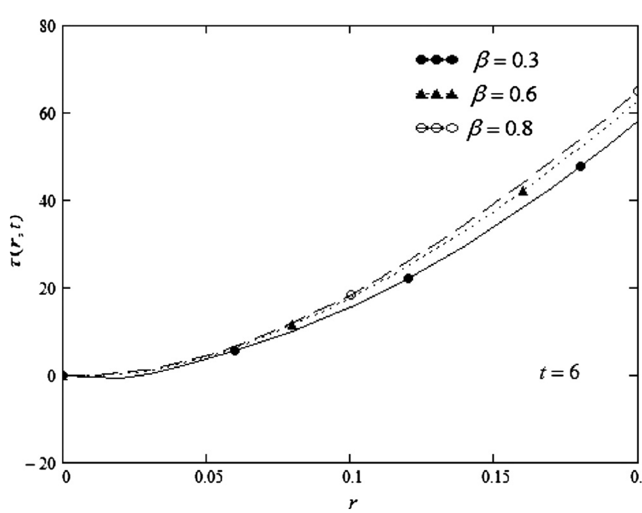

Figure 8 Shear stress $\tau(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\alpha = 0.2$, $t = 6$ sec and different values of β.

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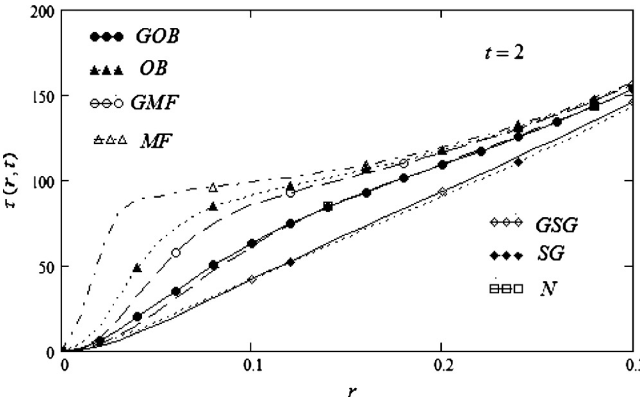

Figure 9 Comparison of shear stress among different fluids $\tau(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$ and at $t = 2$ sec.

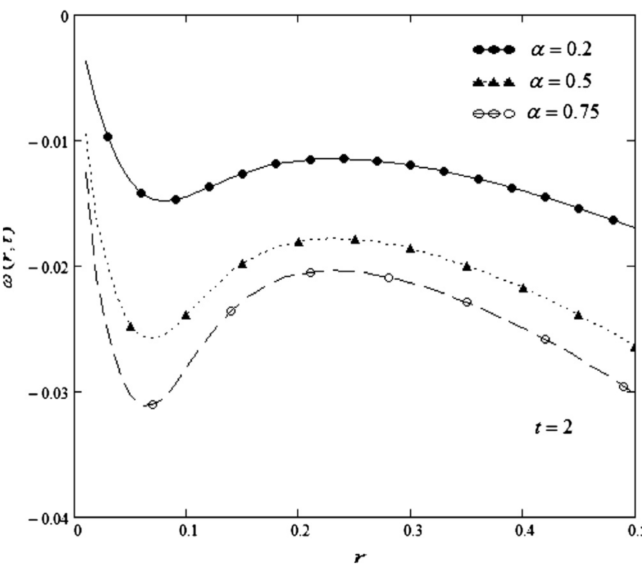

Figure 10 Profiles of velocity $\omega(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\beta = 0.8$, $t = 2$ sec and different values of α.

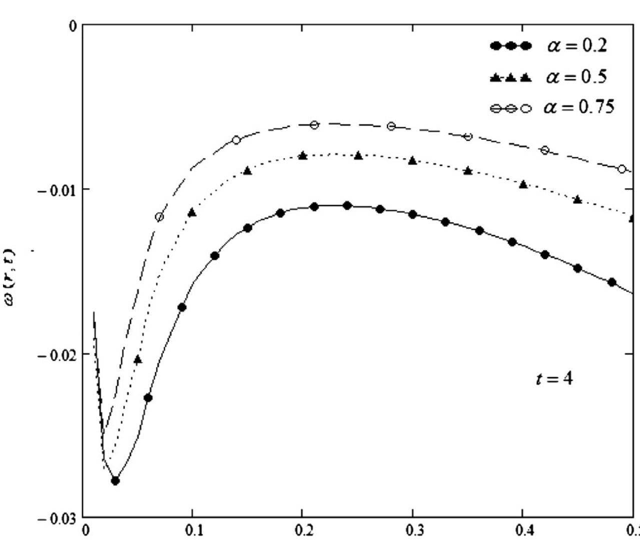

Figure 11 Profiles of velocity $\omega(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\beta = 0.8$, $t = 4$ sec and different values of α.

non-trivial shear stress. The solutions that have been obtained satisfy all imposed initial and boundary conditions and can easily be reduced to the similar solutions corresponding to ordinary Oldroyd-B, fractional/ordinary Maxwell, fractional/ordinary second-grade, and Newtonian fluids performing the similar motion. The obtained results are in the form of Newtonian and non-Newtonian parts. To study the influence of fractional parameters on the velocity and shear stress is plotted in the Figs. 3–15. From Figs. 3–5 it is observed that the $\tau(r,t)$ increases with the increase in value of α and same behavior is observed for β cleared from Figs. 6–8. Figs. 10–15 show that the $\omega(r,t)$ increases with the increase in the values of α and β. Finally, a comparison for time derivative of integer order versus fractional order is shown graphically for GOB, OB,

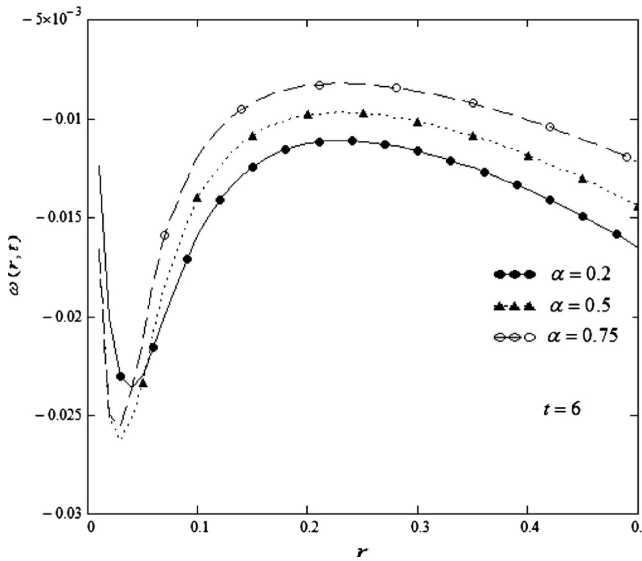

Figure 12 Profiles of velocity $\omega(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\beta = 0.8$, $t = 6$ sec and different values of α.

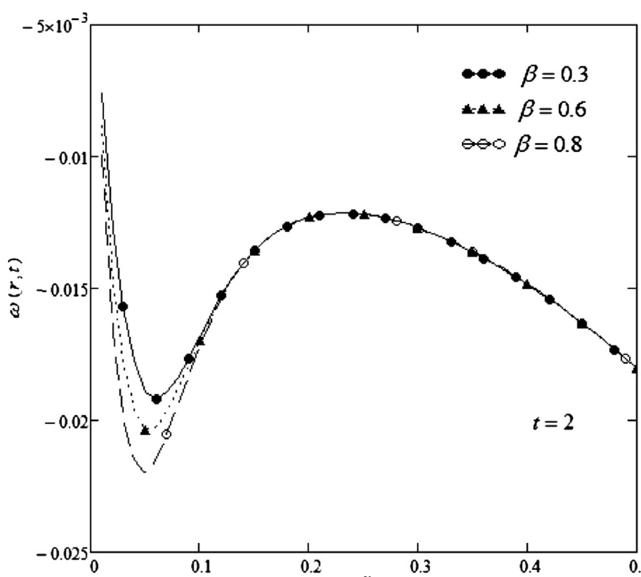

Figure 13 Profiles of velocity $\omega(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\alpha = 0.2$, $t = 2$ sec and different values of β.

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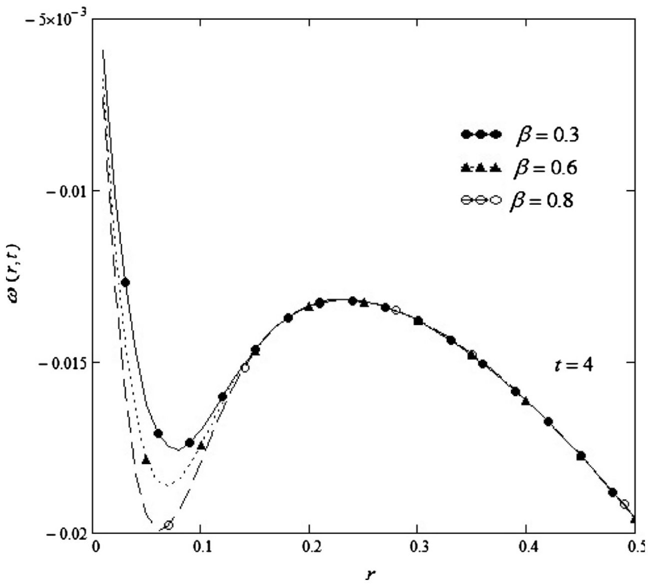

Figure 14 Profiles of velocity $\omega(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\alpha = 0.2$, $t = 4$ sec and different values of β.

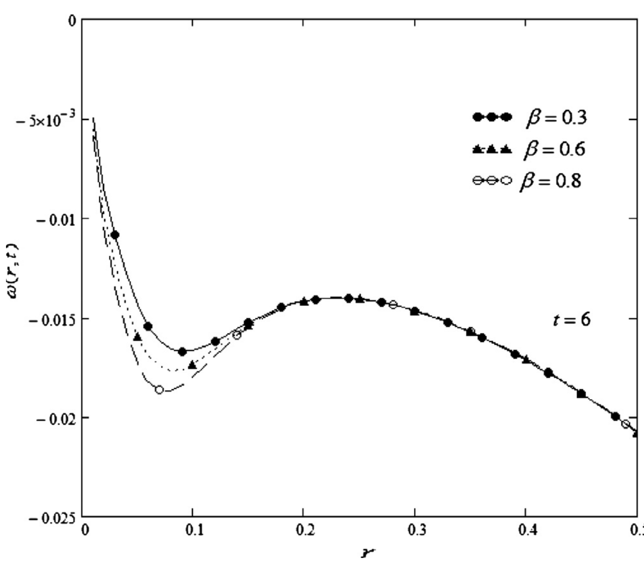

Figure 15 Profiles of velocity $\omega(r,t)$ versus r with $\lambda_r = 0.15$, $\lambda = 0.85$, $\alpha = 0.2$, $t = 6$ sec and different values of β.

GMF, MF, GSG, SG and N in Fig. 9. It is found that ordinary Maxwell fluid is swiftest and ordinary second grade fluid is slowest.

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References

[1] G.K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, Cambridge, 1967.

[2] T.W. Ting, Arch, certain non-steady flows of second-order fluids, Rational Mech. Anal. 14 (1) (1963) 1–26.

[3] P.N. Srivastava, Non-steady helical flow of a visco-elastic liquid, Arch. Mech. Stos. 18 (1966) 145–150.

[4] N.D. Waters, M.J. King, Unsteady flow of an elastico-viscous liquid in a straight pipe of circular cross-section, J. Phys. D: Appl. Phys. 4 (1971) 204–211.

[5] K.R. Rajagopal, R.K. Bhatnagar, Exact solutions for some simple flows of an Oldroyd-B fluid, Acta Mech. 113 (1) (1995) 233–239.

[6] W.P. Wood, Transient viscoelastic helical flows in pipes of circular and annular cross-section, J. Non-Newtonian Fluid Mech. 100 (2001) 115–126.

[7] C. Fetecau, Analytic solutions for non-Newtonian fluid flows in pipe-like domains, Int. J. Nonlinear Mech. 39 (2004) 225–231.

[8] T. Hayat, M. Khan, T. Wang, Non-Newtonian flow between concentric cylinders, Commun. Nonlinear Sci. Numer. Simul. 11 (2006) 297–305.

[9] C. Fetecau, Corina Fetecau, D. Vieru, On some helical flows of Oldroyd-B fluids, Acta Mech. 189 (2007) 53–63.

[10] C. Fetecau, S.C. Prasad, K.R. Rajagopal, A note on the flow induced by a constantly accelerating plate in an Oldroyd-B fluid, Appl. Math. Modell. 31 (2007) 647–654.

[11] Corina Fetecau, T. Hayat, C. Fetecau, Starting solutions for oscillating motions of Oldroyd-B fluids, J. Non-Newtonian Fluid Mech. 153 (2008) 191–201.

[12] S. McGinty, S. McKee, R. McDermott, Analytic solutions of Newtonian and non-Newtonian pipe flows subject to a general time-dependent pressure gradient, J. Non-Newtonian Fluid Mech. 162 (2009) 54–77.

[13] A. Khan, G. Zaman, G. Rahman, Hydromagnetic flow near a non-uniform accelerating plate in the presence of magnetic field through porous medium, J. Porous Media 18 (8) (2015) 801–809.

[14] M. Kamran, M. Imran, M. Athar, M.A. Imran, On the unsteady rotational flow of fractional Oldroyd-B fluid in cylindrical domains, Meccanica 47 (3) (2012) 573–584.

[15] M. Jamil, N.A. Khan, M.I. Asjad, Unsteady rotating flows of Oldroyd-B fluids with fractional derivatives, Int. J. Chem. Reactor Eng. 9 (2011). Article A115.

[16] C. Fetecau, W. Akhtar, M.A. Imran, D. Vieru, On the oscillating motion of an Oldroyd-B fluid between two infinite circular cylinders, Comput. Math. Appl. 59 (8) (2010) 2836–2845.

[17] D. Vieru, Corina Fetecau, M.A. Imran, Unsteady flow of a generalized Oldroyd-B fluid in an oscillating duct of rectangular cross-section, Bul. Inst. Polit. Iasi, Tomul LVI (LX), Fasc. 4 (2010) 47–62.

[18] N. Shahid, M. Rana, M.A. Imran, Flow of an Oldroyd-B fluid over an infinite plate subject to a time dependent shear stress, J. P. Res. Math. 7 (2011) 52–62.

[19] A. Khan, G. Zaman, The motion of generalized Oldroyd-B fluid between two sidewalls of a plate, South Asian J. Math. 5 (2) (2015) 42–52.

[20] M. Renardy, Rocky Mount, Inflow boundary condition for steady flow of viscoelastic fluids with differential constitutive laws, J. Math. 18 (1988) 445–453.

[21] M. Renardy, An alternative approach to inflow boundary conditions for Maxwell fluids in three space dimensions, J. Non-Newtonian Fluid Mech. 36 (1990) 419–425.

[22] R. Talhouk, Unsteady flows of viscoelastic fluids with inflows and outflows boundary condititions, Appl. Math. Lett. 9 (5) (1996) 93–98.

[23] K.R. Rajagopal, A new development and interpretation of the Navier-Stokes fluid which reveals why the Stokes assumption is inapt, Int. J. Non-Linear Mech. 50 (2013) 141–151.

[24] C. Fetecau, K. Kannan, A note on an unsteady flow of an Oldroyd-B fluid, Int. J. Math. Math. Sci. 19 (2005) 3185–3194.

[25] D. Tong, Y. Liu, Exact solutions for the unsteady rotational flow of nonNewtonian fluid in an annular pipe, Int. J. Eng. Sci. 43 (2005) 281–289.

[26] Corina Fetecau, M. Imran, C. Fetecau, I. Burdujan, Helical flow of an Oldroyd-B fluid due to a circular cylinder subject to time dependent shear stresses, Z. Angew. Math. Phys. 61 (2010) 959–969.

[27] M. Jamil, C. Fetecau, M. Imran, Unsteady helical flows of Oldroyd-B fluid, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 1378–1386.

[28] Corina Fetecau, M. Imran, C. Fetecau, Taylor-Couette flow of an Oldroyd-B fluid in an annulus due to a time-dependent couple, Zeit. Nat. A 66 (a) (2011) 40–46.

[29] A.A. Zafar, A. Rauf, D. Vieru, On exact solution for flow of fractional Oldroyd-B fluid betweentwo side walls prependicularto a plate, Bull. Inst. Polit. Iasi, Tomul LVIII (LXII), Fasc. (2012) 218–229.

[30] Corina Fetecau, M. Rana, N. Nigar, C. Fetecau, First exact solutions for flows of rate type fluids in a circular duct that applies a constant couple to the fluid, Z. Naturforsch. 69 (a) (2014) 232–238.

[31] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[32] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA, 1999.

[33] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach Sci. Publishers, Yverdon, 1993.

[34] I.N. Sneddon, Fourier Transforms, McGraw-HillBook Company, Inc., New York, Toronto, London, 1951.

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