Fourth-Order Method For The Solution Of Diffusion Equation Subject To The Specification Of Energy

M. A. Rehman (Corresponding author)
Department of Mathematics, University of Management and Technology, C-II Johar Town, Lahore, Pakistan
Tel: +92-333-4481003 Email: aziz.rehman@umt.edu.pk
M. S. A. Taj
Department of Mathematics, COMSATS Institute of information Technology, Defence Road, Off Raiwind Road, Lahore, Pakistan
Tel: +92-300-4417364 Email: shahadat259@yahoo.com
M. Saeed
Department of Mathematics, University of Management and Technology, C-II Johar Town, Lahore, Pakistan
Tel: +92-300-5333043 Email: muhammad.saeed@umt.edu.pk
I. Rehman
Department of Mathematics, COMSATS Institute of information Technology, Abbattabad, Pakistan
Tel: +92-300-5135457 Email: sinayat@ciit.net.pk

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Abstract

A fourth-order numerical technique is developed for the solution of the diffusion equation $u_t = u_{xx} + s(x,t), 0 < x < X, 0 < t \le T$, subject to $u(x,0) = f(x), 0 < x < X, u(1,t) = g(t), 0 < t \le T$ and the specification of energy $\int_0^b u(x,t)dx = M(t), 0 < b < X, 0 < t \le T$.

Keywords: Diffusion equation, non-local boundary condition, specification of energy, fourth-order method, method of lines.

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1. Introduction

In the last two decades, the development of the numerical techniques for the solution of the non-local boundary value problems has been an important research topic of science and engineering. Particularly thermo-elasticity has been the subject of some resent works [4, 5].

Many physical phenomenon are modeled by non-classical boundary value problems with non-local boundary conditions. One could generally classify these into two types: (i) boundary value problems with nonlocal initial condition or (ii) boundary value problems with nonlocal boundary conditions. The present work focuses on the second group of these nonlocal boundary value problems.

In this paper we have considered the diffusion equation with a nonlocal boundary condition: the so called energy specification . This is a linear constant having the form $\int_0^b u(x,t)dx = E(t)$ where b is a constant and E(t) is the given function. Along with a one dimensional parabolic equation, this condition is quite different from the classical boundary condition. Nonlocal boundary value problems have certainly been one of the fastest growing areas in various application fields. Science and industry are both responsible for this growth.

Certain problems arising in the thermodynamics, heat condition, plasma physics can be reduced to the nonlocal problems with integral condition. Mathematical formation of this kind also arises in the transport of reactive and passive contaminates in aquifer, an area of active interdisciplinary research of mathematics. We refer the reader to [2] for the derivation of the mathematical models and for the precise hypothesis and analysis.

The authors of [7] have given an example from metrology. This example is the method for the evaluation of the temperature distribution of the air near the ground during the calm clear nights.

One very common characteristic of all these models is that they all express a conservation of certain quantity (mass, momentum, heat, etc) in any moment for any sub domain. This in many applications is the most desirable feature of the approximation method when it comes to the solution of the corresponding initial value problem.

Much attention has been paid in the literature for the development, analysis and implementation of accurate methods for the numerical solution of the time dependent partial differential equations with nonlocal boundary condition.

This paper consider the problem of obtaining numerical approximation to the concentration $u = u(x,t)$ which satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + s(x,t),\ 0 < x < X,\ 0 < t \le T \qquad (1)$$

subject to the given initial condition

$$u(x,0) = f(x),\ 0 < x < X \qquad (2)$$

with boundary condition

$$u(1,t) = g(t), 0 < t \le T \qquad (3)$$

and the nonlocal boundary condition

$$\int_0^b u(x,t)dx = M(t),\ 0 < b < X,\ 0 < t \le T \qquad (4)$$

where f, g, M and s are known functions and are assumed to be sufficiently smooth to produce a smooth classical solution of u.

A number of sequential numerical procedures have been suggested in the literature for the solution of this problem: see, for instant, [1] and [3].

In the present paper the method of lines semi discretization approach will be used to transform the model partial differential equation (PDE) into a system of first order, linear, ordinary differential equations (ODE's), the solution of which satisfies a certain recurrence relation involving matrix exponential terms. A suitable rational approximation will be used to approximate such exponentials leading to an L_o-stable algorithm which may be parallelized through the partial fraction splitting technique.

2. Discritization and recurrence relation

Choosing a positive integer $N \geq 6$ and dividing the interval $[0, X]$ into $N+1$ subintervals each of width h, so that $(N+1)h = X$, and the time variable t into time steps each of length l gives a rectangular mesh of points with co-ordinates $(x_m, t_n) = (mh\,, nl)$ $(m = 0,1,2, \dots, N, N+1)$ and $(n = 0,1,2, \dots)$ covering the region $R = [0 < x < X] \times t$ and its boundary ∂R consisting of the lines $x = 0,\; x = X$ and $t = 0$.

Assuming that $u(x,t)$ is six time continuously differentiable with respect to variable x and that these derivatives are uniformly bounded, the space derivative in (1) may be approximated to the fourth order accuracy at some general point (x,t) of the mesh by using five point formula:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{12\,h^2}\{-\mathrm{u(x-2h,t)} + 16\mathrm{u(x-h,t)} - 30\mathrm{u(x,t)} + 16\mathrm{u(x+h,t)} - \mathrm{u(x+2h,t)}\} + \frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6} + \mathrm{O(h^5)}\ as\ h \to 0 \qquad (5)$$

It is worth noting that the equation (5) is valid only for $(\mathrm{x,t}) = (x_m, t_n)$ with $m = 2,3, \dots, N-1$. To attain the accuracy at the same points at (x_1, t_n) and (x_n, t_n) special formulae must be developed which approximate $\frac{\partial^2 u(x,t)}{\partial x^2}$ not only to fourth-order but also with dominant error term $\frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6}$. It can be clearly shown that such approximations will be

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{12\,h^2}\{9\mathrm{u(x-h,t)} - 9\mathrm{u(x,t)} - 19\mathrm{u(x+h,t)} + 34\mathrm{u(x+2h,t)} - 21\,\mathrm{u(x+3h,t)} + 7\,\mathrm{u(x+4h,t)} - \mathrm{u(x+5h,t)}\} + \frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6} + \mathrm{O(h^5)}\ as\ h \to 0 \qquad (6)$$

and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{12\,h^2}\{-\mathrm{u(x-5h,t)} + 7\mathrm{u(x-4h,t)} - 21\mathrm{u(x-3h,t)} + 34\mathrm{u(x-2h,t)}$$

$$-19\,\mathrm{u(x-h,t)} - 9\,\mathrm{u(x,t)} + 9\,\mathrm{u(x-h,t)}\} + \frac{h^4}{90}\frac{\partial^6 u(x,t)}{\partial x^6} + \mathrm{O(h^5)}\ as\ h \to 0 \quad (7)$$

at the mesh points (x_1, t_n) and (x_n, t_n) respectively.

3. Treatment of nonlocal boundary condition

The integral in (4) may be approximated by using a quadrature rule such as Simpsom's 1/3 rule to give us

$$\int_0^b u(x,t)\,dx \approx \frac{h^*}{3}\left[u(0,t) + 4\sum_{i=1}^{\frac{J}{2}} u(2i-1,t) + 2\sum_{i=1}^{\frac{J}{2}-1} u(2i,t) + u(J,t)\right] + O(h^4) \qquad (8)$$

In which $h^* = \dfrac{b}{J}$. Thus (8) serves as the boundary condition at zero. Appling (1) with (5), (6)and (7) to all the mesh points of the grid at time level $t = t_n$ produces a system of ODE's of the form

$$\frac{d\boldsymbol{U}(t)}{dt} = A\,\boldsymbol{U}(t) + v(t), \;\; t > 0 \qquad (9)$$

With initial distribution

$$\boldsymbol{U}(0) = f \qquad (10)$$

in which $\boldsymbol{U}(t) = [\,U_1(t),\, U_2(t), \dots,\, U_N(t)]^T$, $f = [f(x_1), f(x_2), \dots, f(x_N)]^T$, T denotes transpose and the matrix A is of order N which is given by

$$A = \frac{1}{12h^2}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \dots & \alpha_J & & \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \dots & \beta_J & & \\ -1 & 16 & -30 & 16 & -1 & & & & \\ & -1 & 16 & -30 & 16 & -1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & -1 & 16 & -30 & 16 & -1 \\ & & & & & -1 & 16 & -30 & 16 \\ & & & -1 & 7 & -21 & 34 & -19 & -9 \end{bmatrix}$$

where $\alpha_1 = -45,\ \alpha_2 = -37, \alpha_3 = -2,\ \alpha_4 = -39, \alpha_5 = -29, \alpha_6 = -19$

and $$\alpha_i = \begin{cases} -36 & for \;\; i = 7(2)J-1 \\ -18 & for \;\; i = 8(2)J-2 \\ -9 & for \;\; i = J \end{cases}$$

also $\beta_1 = 20,\ \beta_2 = -28, \beta_3 = 20,\ \beta_4 = 1$

and $$\beta_i = \begin{cases} 4 & for \;\; i = 5(2)J-1 \\ 2 & for \;\; i = 6(2)J-2 \\ 1 & for \;\; i = J \end{cases}$$

the column vector $v(t)$ contains the contribution from the functions $M(t)$ and $g(t)$ The solution of the system (9) and (10) satisfies the recurrence relation

$$\boldsymbol{U}(t+l) = \exp(lA)\, f + \int_0^t \exp\big((t-s)A\big) v(s)\, ds; \quad t \geq 0 \tag{11}$$

which satisfies the recurrence relation

$$\boldsymbol{U}(t+l) = \exp(lA)\, \boldsymbol{U}(t) + \int_t^{t+l} \exp\big((t+l-s)A\big) v(s)\, ds, \quad t = 0, l, 2l, \ldots \tag{12}$$

The eigenvalues of the matrix *A* are not in closed form. The value of *N* must be chosen so that eigenvalues are distinct with negative real part see [8], this can be tested by using, for example, Matlab etc.

To approximate the matrix exponential in (12) we use rational approximation [9] consisting of three parameters and given for the real scalar θ by

$$E_4(\theta) = \frac{1 + (1-a_1)\theta + \left(\frac{1}{2} - a_1 + a_2\right)\theta^2 + \left(\frac{1}{6} - \frac{a_1}{2} + a_2 - a_3\right)\theta^3}{1 - a_1\theta + a_2\theta^2 - a_3\theta^3 + \left(-\frac{1}{24} + \frac{a_1}{6} - \frac{a_2}{2} + a_3\right)\theta^4} = \frac{p(\theta)}{q(\theta)} \tag{13}$$

with error constant $C_5 = \frac{1}{30} - \frac{1}{8}a_1 + \frac{1}{3}a_2 - \frac{1}{2}a_3$ [8].

Let λ be the eigenvalue of the matrix A then the amplification symbol of the numerical method arising from (13) is

$$R(-z) = \frac{1 - (1-a_1)z + \left(\frac{1}{2} - a_1 + a_2\right)z^2 - \left(\frac{1}{6} - \frac{a_1}{2} + a_2 - a_3\right)z^3}{1 + a_1 z + a_2 z^2 + a_3 z^3 + \left(-\frac{1}{24} + \frac{a_1}{6} - \frac{a_2}{2} + a_3\right)z^4} \tag{14}$$

Where $z = -l\,\mathrm{Re}(\lambda) > 0$. It can be shown that the resulting method is L-stable provided that

$$\left.\begin{array}{l} a_1 > \frac{1}{2} \\ a_2 > \frac{a_1}{2} - \frac{1}{6} \\ a_3 > \frac{a_2}{2} - \frac{a_1}{6} + \frac{1}{24} \end{array}\right\} \tag{15}$$

So we have

$$\exp(lA) = G^{-1}\left(I + (1-a_1)\, lA + \left(\frac{1}{2} - a_1 + a_2\right) l^2A^2 + \left(\frac{1}{6} - \frac{a_1}{2} + a_2 - a_3\right) l^3A^3\right) \tag{16}$$

where

$$G = 1 - a_1 lA + a_2\, l^2A^2 - a_3 l^3A^3 + \left(-\frac{1}{24} + \frac{a_1}{6} - \frac{a_2}{2} + a_3\right) l^4A^4 \tag{17}$$

We have chosen $a_1 = \frac{64}{25}$, $a_2 = \frac{7}{3}$ and $a_3 = \frac{547}{600}$ it can be shown that using these values L-stability is granted [10].

The quadrature term in (12) will be approximated by

$$\int_t^{t+l} \exp\big((t+l-s)A\big) v(s)\, ds = W_1 v(s_1) + W_2 v(s_2) + W_3 v(s_3) + W_4 v(s_4) \tag{18}$$

where $s_1 \neq s_2 \neq s_3 \neq s_4$ and $W_i (i = 1,2,3,4)$ are matrices.
It can be shown that

(i) When $v(s) = [1,1,1,...,1]^T$

$$W_1 + W_2 + W_3 + W_4 = M_1 \quad (19)$$

Where $M_1 = A^{-1}(\exp(lA) - I)$

(ii) When $v(s) = [s, s, s, ..., s]^T$

$$s_1 W_1 + s_2 W_2 + s_3 W_3 + s_4 W_4 = M_2 \quad (20)$$

Where $M_2 = A^{-1}[t\exp(lA) - (t+l)I + A^{-1}(\exp(lA) - I)]$

(iii) When $v(s) = [s^2, s^2, s^2, ..., s^2]^T$

$$s_1^2 W_1 + s_2^2 W_2 + s_3^2 W_3 + s_4^2 W_4 = M_3 \quad (21)$$

Where $M_3 = A^{-1}[t^2 \exp(lA) - (t+l)^2 I + 2A^{-1}\{t\exp(lA) - (t+l)I + A^{-1}(\exp(lA) - I)\}]$

(iv) When $v(s) = [s^3, s^3, s^3, ..., s^3]^T$

$$s_1^3 W_1 + s_2^3 W_2 + s_3^3 W_3 + s_4^3 W_4 = M_4 \text{ where} \quad (22)$$

$$M_4 = A^{-1}[t^3 \exp(lA) - (t+l)^3 I + 3A^{-1}[t^2 \exp(lA) - (t+l)^2 I + 2A^{-1}\{t\exp(lA) - (t+l)I + A^{-1}(\exp(lA) - I)\}]]$$

Taking $s_1 = t,\ s_2 = t + \frac{l}{3},\ s_3 = t + \frac{2l}{3}, s_4 = t + l$ and then solving (19)-(22) simultaneously gives

$$W_1 = \frac{9}{2l^3}(A^{-1})^4\{6I + 2lA + \frac{2}{9}l^2A^2 - (6I - 4lA + \frac{11}{9}l^2A^2 - \frac{2}{9}l^3A^3)\exp(lA)\} \quad (23)$$

$$W_2 = -\frac{27}{2l^3}(A^{-1})^4\{6I + \frac{8}{3}lA + \frac{1}{3}l^2A^2 - (6I - \frac{10}{3}lA + \frac{2}{3}l^2A^2)\exp(lA)\} \quad (24)$$

$$W_3 = \frac{27}{2l^3}(A^{-1})^4\{6I + \frac{10}{3}lA + \frac{2}{3}l^2A^2 - (6I - \frac{8}{3}lA + \frac{1}{3}l^2A^2)\exp(lA)\} \quad (25)$$

$$W_4 = -\frac{9}{2l^3}(A^{-1})^4\{6I + 4lA + \frac{11}{9}l^2A^2 + \frac{2}{9}l^3A^3 - (6I - 2lA + \frac{2}{9}l^2A^2)\exp(lA)\} \quad (26)$$

Now by replacing *exp(lA)* by (16) and (17) in (23)-(26)

$$W_1 = \frac{l}{24}\{3I - (19 - 78a_1 + 216a_2 - 324a_3)lA + (3 - 8a_1 + 12a_2)l^2A^2\}G^{-1} \quad (27)$$

$$W_2 = \frac{3l}{16}\{2I + (16 - 56a_1 + 144a_2 - 216a_3)lA + (1 - 4a_1 + 12a_2 - 24a_3)l^2A^2\}G^{-1} \quad (28)$$

$$W_3 = \frac{3l}{8}\{I - (7 - 26a_1 + 72a_2 - 108a_3)lA - (1 - 4a_1 + 12a_2 - 24a_3)l^2A^2\}G^{-1} \quad (29)$$

$$W_4 = \frac{l}{48}\{6I + (44 - 168a_1 + 432a_2 - 648a_3)lA + (11 - 44a_1 + 132a_2 - 216a_3)l^2A^2 + (2 - 8a_1 + 24a_2 - 48a_3)l^3A^3\}G^{-1} \quad (30)$$

Hence (12) can be written as

$$U(t+l) = \exp(lA)U(t) + W_1 v(t) + W_2 v(t + \frac{l}{3}) + W_3 v(t + \frac{2l}{3}) + W_4 v(t+l) \quad (31)$$

Where W_1, W_2, W_3 *and* W_4 are given by (27)-(30).

4. Development of Algorithm

Assuming a_1, a_2 and a_3 satisfy the conditions (15) to produce real and distinct zeros $r_i (i = 1,2,3,4)$ $r_i \neq 0$ for $q(\theta)$ of (13) , then

$$G = (I - \frac{l}{r_1}A)(I - \frac{l}{r_2}A)(I - \frac{l}{r_3}A)(I - \frac{l}{r_4}A) \quad (32)$$

and

$$\exp(lA) = \sum_{j=1}^{4} p_j \, (I - \frac{l}{r_j}A)^{-1} \quad (33)$$

where $p_j (j = 1,2,3,4)$, the partial fraction coefficient of $E_4(\theta)$, are defined by

$$p_j = \frac{\left\{1+(1-a_1)r_j+\left(\frac{1}{2}-a_1+a_2\right)r_j^2+\left(\frac{1}{6}-\frac{a_1}{2}+a_2-a_3\right)r_j^3\right\}}{\prod_{\substack{i=1 \\ i\neq j}}^{4}(1-\frac{r_j}{r_i})}, \quad j = 1,2,3,4 \quad (34)$$

$$\text{and } W_k = \frac{m_k l}{48} \sum_{j=1}^{4} p_{4k+j} \, (I - \frac{l}{r_j}A)^{-1}, \qquad k = 1,2,3,4 \quad (35)$$

in which $m_1 = 2, m_2 = 9, m_3 = 18, m_4 = 1$ and for j=1,2,3,4

$$p_{4+j} = \frac{3 + (19 - 78a_1 + 216a_2 - 324a_3)r_j \;\; + (3 - 8a_1 + 12a_2)r_j^2}{\prod_{\substack{i=1 \\ i\neq j}}^{4}(1 - \frac{r_j}{r_i})}$$

$$p_{8+j} = \frac{2 + (16 - 56a_1 + 144a_2 - 216a_3)r_j \;\; + (1 - 4a_1 + 12a_2 - 24a_3)r_j^2}{\prod_{\substack{i=1 \\ i\neq j}}^{4}(1 - \frac{r_j}{r_i})}$$

$$p_{12+j} = \frac{1 + (-7 + 26a_1 - 72a_2 + 108a_3)r_j \;\; + (1 - 4a_1 + 12a_2 - 24a_3)r_j^2}{\prod_{\substack{i=1 \\ i\neq j}}^{4}(1 - \frac{r_j}{r_i})}$$

$$p_{16+j} = \frac{1}{\prod_{\substack{i=1 \\ i\neq j}}^{4}(1 - \frac{r_j}{r_i})} \times \left\{ \begin{matrix} 6 + (44 - 168a_1 + 432a_2 - 648a_3)r_j \;\; + (11 - 44a_1 + 132a_2 - 216a_3)\, r_j^2 + \\ (2 - 8a_1 + 24a_2 - 48a_3)r_j^3 \end{matrix} \right\}$$

Hence

$$\exp(lA)\, U(t) = \left[p_1 \left(I - \frac{l}{r_1}A\right)^{-1} + p_2 \left(I - \frac{l}{r_2}A\right)^{-1} + p_3 \left(I - \frac{l}{r_3}A\right)^{-1} + p_4 \left(I - \frac{l}{r_4}A\right)^{-1}\right] U(t) \quad (36)$$

The implementation of the method using a parallel architecture with four processors is based on the partial decomposition [6] of $\exp(lA)U(t), W_1 v(t), W_2 v(t+\frac{l}{3}), W_3 v(t+\frac{2l}{3})$ *and* $W_4 v(t+l)$ in (31).

Hence

$$\boldsymbol{U}(t+l) = \sum_{i=1}^{4} A_i^{-1} \left[p_i \boldsymbol{U}(t) + \frac{l}{48}\left\{2p_{i+4}v(t) + 9\, p_{i+8}v\left(t+\frac{l}{3}\right) + 18p_{i+12}v\left(t+\frac{2l}{3}\right) + p_{i+16}v(t+l)\right\}\right] \quad (37)$$

where

$$A_i = I - \frac{l}{r_i} A_i \,, i = 1,2,3,4$$

so

$$\boldsymbol{U}(t+l) = \sum_{i=1}^{4} y_i(t)$$

where $y_i, i = 1,2,3,4$ are the solutions of the system

$$A_i y_i = p_i U(t) + \frac{l}{48}\left\{2p_{i+4}v(t) + 9\,p_{i+8}v\left(t+\frac{l}{3}\right) + 18p_{i+12}v\left(t+\frac{2l}{3}\right) + p_{i+16}v(t+l)\right\} \qquad (38)$$

respectively.

5 The Parallel Algorithm

Equations (38) have great importance in the parallel environment since they can be used to solve the corresponding linear algebraic systems on processors operating concurrently. $U(t+l)$ in (31), the solution vector at time $t=(n+1)l$, may be obtained via the parallel algorithm using four different processors in the following form:

5.1 Processor 1

Step 1:	$l, r_1, p_1, p_5, p_9, p_{13}, p_{17}, \boldsymbol{U}_o, A$
Step 2:	Compute $I - \frac{l}{r_1} A$
Step 3:	Decompose $I - \frac{l}{r_1} A = L_1 U_1$
Step 4:	Evaluate $v(t), v\left(t+\frac{l}{3}\right), v\left(t+\frac{2l}{3}\right)$ and $v(t+l)$,
Step 5:	Using $W_1(t) = 2p_5 v(t) + 9p_9\, v\left(t+\frac{l}{3}\right) + 18p_{13}v\left(t+\frac{2\,l}{3}\right) + p_{17}v(t+l)$
Step 6:	Solve $L_1U_1\ \ y_1(t) = p_1\boldsymbol{U}(t) + \frac{l}{48}W_1(t)$

5.2 Processor 2

Step 1:	$l, r_2, p_2, p_6, p_{10}, p_{14}, p_{18}, \boldsymbol{U_o}, A$
Step 2:	Compute $I - \frac{l}{r_2} A$
Step 3:	Decompose $I - \frac{l}{r_2} A = L_2 U_2$
Step 4:	Evaluate $v(t), v\left(t+\frac{l}{3}\right), v\left(t+\frac{2l}{3}\right)$ and $v(t+l)$,
Step 5:	Using $W_2(t) = 2p_6 v(t) + 9p_{10}\, v\left(t+\frac{l}{3}\right) + 18p_{14}v\left(t+\frac{2\,l}{3}\right) + p_{18}v(t+l)$
Step 6:	Solve $L_2U_2\ \ y_2(t) = p_2\boldsymbol{U}(t) + \frac{l}{48}W_2(t)$

5.3 Processor 3

Step 1:	$l, r_3, p_3, p_7, p_{11}, p_{15}, p_{19}, \boldsymbol{U_o}, A$
Step 2:	Compute $I - \frac{l}{r_3}A$
Step 3:	Decompose $I - \frac{l}{r_3}A = L_3U_3$
Step 4:	Evaluate $v(t), v\left(t+\frac{l}{3}\right), v\left(t+\frac{2l}{3}\right)$ and $v(t+l)$,
Step 5:	Using $W_3(t) = 2p_7v(t) + 9p_{11}\, v\left(t+\frac{l}{3}\right) + 18p_{15}v\left(t+\frac{2\,l}{3}\right) + p_{19}v(t+l)$
Step 6:	Solve $L_3U_3\ \ y_3(t) = p_3\boldsymbol{U}(t) + \frac{l}{48}W_3(t)$

5.4 Processor 4

Step 1:	$l, r_4, p_4, p_8, p_{12}, p_{16}, p_{20}, \boldsymbol{U_o}, A$
Step 2:	Compute $I - \frac{l}{r_4}A$
Step 3:	Decompose $I - \frac{l}{r_4}A = L_4U_4$
Step 4:	Evaluate $v(t), v\left(t+\frac{l}{3}\right), v\left(t+\frac{2l}{3}\right)$ and $v(t+l)$,
Step 5:	Using $W_4(t) = 2p_8v(t) + 9p_{12}\, v\left(t+\frac{l}{3}\right) + 18p_{16}v\left(t+\frac{2\,l}{3}\right) + p_{20}v(t+l)$
Step 6:	Solve $L_4U_4\ \ y_4(t) = p_4\boldsymbol{U}(t) + \frac{l}{48}W_4(t)$

Then solve $\boldsymbol{U}(t+l) = \sum_{i=1}^{4} y_i(t)$.

In implementing these algorithms, processor (1) generates the decomposition of $I - \frac{l}{r_1}A$, anprocessors (2) generates the decomposition of $I - \frac{l}{r_2}A$ while processor (3) generates the decomposition of $I - \frac{l}{r_3}A$ and processor (4) generates decomposition of $I - \frac{l}{r_4}A$ simultaneously.

6 Numerical Experiments

In this section the numerical method described in this paper will be applied to two problems from the literature and results obtained will be compared with the results obtained by the method already existing in the literature.

6.1 Example (1)

Consider (1), (2), (3) and (4) with

$$f(x) = \exp(x),\ 0 < x < 1$$
$$g(t) = \frac{e}{1+t^2},\ 0 < t < 1$$
$$b = 0.3$$
$$M(t) = \frac{e^{0.3}-1}{1+t^2}, 0 < t \le 1$$
$$s(x,t) = \frac{-(1+t)^2 \exp(x)}{(1+t^2)^2}, 0 < t \le 1, 0 < x < 1$$

and with theoretical solution

$$u(x,t) = \frac{\exp(x)}{1+t^2}$$

The results of the $u(x,t)$ with h=0.01, 0.005, 0.0025, 0.001 and $l = 0.01, 0.005, 0.0025, 0.001$, using the fourth-order scheme discussed in this paper are shown in Table (1) and are compared with the result obtained by using the implicit finite- difference scheme of [1] and the pade scheme of [3].

In the Table (1), we have presented the relative error, $\frac{|U_{approx} - U_{exact}|}{U_{exact}}$, for our formula and the methods of [1] and [3].

The result obtained using the fourth order scheme developed in this paper are highly accurate than those of [1] and [3]. It is also noted that the method developed in this paper is fourth order accurate but only for small values of *l* and *h* it deviates slightly. It is therefore clear that as far as efficiency is concerned, the fourth order scheme introduced in this paper is the best candidate for the model problem. This technique can also be coded very efficiently on the super as well as parallel computers.

6.2 Example (2)

Consider (1), (2), (3) and (4) with

$$f(x) = \sin(\pi x),\ 0 < x < 1$$
$$g(t) = 0,\ 0 < t < 1$$
$$b = 0.75,$$
$$M(t) = \frac{2+\sqrt{2}}{2\pi}\exp(-\pi^2 t), 0 < t \le 1$$
$$s(x,t) = 0,\ 0 < t \le 1, 0 < x < 1$$

and with theoretical solution

$$u(x,t) = \exp(-\pi^2 t)\sin(\pi x).$$

The results for example (6.2) are given in Table (2). Calculations were performed for different values of h=0.01, 0.005, 0.0025, 0.001 and $l = 0.01, 0.005, 0.0025, 0.001$.

The results show that the fourth order scheme developed in this paper gave superior results than that of [1] and [3] and also fourth order accurate. It is worth mentioning that the use of only real arithmetic especially in multi-space dimension can yield with large saving of CPU time used.

Table 1: Maximum error for Example (1)

h	Numerical Method	l = 0.01	l = 0.005	l = 0.0025	l = 0.001

0.01	The implicit scheme The pade scheme Fourth order scheme	8.0×10^{-04} 6.0×10^{-05} 2.7×10^{-09}	2.0×10^{-04} 1.0×10^{-05} 3.3×10^{-10}	6.0×10^{-04} 5.0×10^{-05} 7.4×10^{-11}	2.0×10^{-04} 1.0×10^{-05} 7.4×10^{-11}
0.005	The implicit scheme The pade scheme Fourth order scheme	7.0×10^{-04} 5.0×10^{-05} 2.8×10^{-09}	2.0×10^{-04} 2.0×10^{-05} 1.9×10^{-10}	5.0×10^{-04} 4.0×10^{-05} 4.8×10^{-11}	3.0×10^{-04} 3.0×10^{-05} 5.9×10^{-11}
0.0025	The implicit scheme The pade scheme Fourth order scheme	6.0×10^{-04} 4.0×10^{-05} 2.9×10^{-09}	3.0×10^{-04} 2.0×10^{-05} 2.9×10^{-10}	5.0×10^{-04} 3.0×10^{-05} 1.3×10^{-10}	3.0×10^{-04} 2.0×10^{-05} 1.83×10^{-10}
0.001	The implicit scheme The pade scheme Fourth order scheme	3.0×10^{-04} 1.0×10^{-05} 2.7×10^{-09}	5.0×10^{-04} 3.0×10^{-05} 9.13×10^{-11}	7.0×10^{-05} 4.0×10^{-06} 2.8×10^{-10}	9.0×10^{-05} 5.0×10^{-06} 9.7×10^{-10}

Table 2: Maximum error for Example (2) at *t=1*

h	Numerical Method	l = 0.01	l = 0.005	l = 0.0025	l = 0.001
0.01	The implicit scheme The pade scheme Fourth order scheme	9.0×10^{-04} 6.0×10^{-05} 7.0×10^{-07}	3.0×10^{-04} 2.0×10^{-05} 7.0×10^{-07}	2.0×10^{-04} 1.0×10^{-05} 7.0×10^{-07}	1.0×10^{-04} 1.0×10^{-05} 7.0×10^{-07}
0.005	The implicit scheme The pade scheme Fourth order scheme	8.0×10^{-04} 5.0×10^{-05} 4.9×10^{-10}	4.0×10^{-04} 2.0×10^{-05} 4.7×10^{-11}	5.0×10^{-04} 2.0×10^{-05} 4.1×10^{-12}	4.0×10^{-04} 3.0×10^{-05} 1.1×10^{-13}
0.0025	The implicit scheme The pade scheme Fourth order scheme	7.0×10^{-04} 6.0×10^{-05} 4.9×10^{-10}	5.0×10^{-04} 2.0×10^{-05} 4.7×10^{-11}	4.0×10^{-04} 3.0×10^{-05} 4.2×10^{-12}	2.0×10^{-04} 1.0×10^{-05} 1.7×10^{-13}
0.001	The implicit scheme The pade scheme Fourth order scheme	3.0×10^{-04} 1.0×10^{-05} 4.9×10^{-10}	2.0×10^{-04} 3.0×10^{-05} 4.7×10^{-11}	3.0×10^{-05} 5.0×10^{-06} 4.2×10^{-12}	1.0×10^{-05} 2.0×10^{-06} 1.1×10^{-13}

7. Conclusions

In this paper, an algorithm was applied to the one-dimensional diffusion equation with a nonlocal by replacing one standard boundary value condition. The exact solution of this system of first order ODEs satisfies a recurrence relation involving the matrix exponential function. This function is approximated by a rational function possessing real and distinct poles which consequently readily admits the partial fraction expansion, thereby allowing the distribution of the work in solving the corresponding linear algebraic systems on four concurrent processors. The method developed is fourth order accurate. It does not require the use of complex arithmetic and need only real arithmetic in its implementation. This technique work very well for the one-dimensional diffusion equation with the two existing schemes [1] and [3] in the literature and may be implemented in parallel using a machine with four processors concurrently.

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