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Dynamics of axial symmetric system in self-interacting Brans–Dicke gravity

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Abstract This paper investigates the dynamics of an axial reflection symmetric model in self-interacting Brans–Dicke gravity for anisotropic fluid. We formulate hydrodynamical equations and discuss oscillations using a time-dependent perturbation for both spin-dependent and spin-independent cases. The expressions of the frequency, the total energy density, and the equation of motion of the oscillating model are obtained. We study the instability of the oscillating models in weak approximations. It is found that the oscillations and stability of the model depend upon the dark energy source along with anisotropy and reflection effects. We conclude that the axial reflection system remains stable for stiffness parameter $\Gamma = 1$, collapses for $\Gamma > 1$, and becomes unstable for $0 < \Gamma < 1$.

1 Introduction

Dark energy and stellar evolution are interesting issues of modern cosmology as well as gravitational physics. Various astronomical surveys (such as Sloan Digital Sky Survey, Wilkinson Microwave Anisotropy Probe, Supernova type Ia, large scale-structure, weak lensing and galactic cluster emission of X-rays etc.) reveal accelerated expansion of the universe [1–4]. It is assumed that a mysterious form of energy, termed dark energy, is responsible for this accelerated expansion of the universe. The resolution of this mystery leads to various modified theories of gravity by modifying the Einstein–Hilbert action. In this context, the scalar–tensor theory is one of the most fascinating ideas which has provided solutions of various cosmic problems, such as early and late behavior of the universe, inflation, the coincidence problem, and cosmic acceleration [5–7].

The most explored and useful example of a scalar–tensor gravitational framework is the Brans–Dicke theory of gravity. This is a natural generalization of general relativity, constructed by the coupling of the tensor field R and a massless scalar field ϕ. It also contains a constant tuneable parameter ω_{BD}, which can be tuned according to suitable observations. The concept of this theory is based upon the weak equivalence principle, Mach's principle, and Dirac's large number hypothesis [8–10]. The basic idea of this theory is that the inertial mass of an object is not an intrinsic property of the object itself but is generated by the gravitational effect of all the other matter in the universe. For cosmic inflation, this theory is generalized with self-interacting scalar field by the inclusion of scalar potential function $V(\phi)$ [11–13] known as self-interacting Brans–Dicke (SBD) gravity. This has attracted a community of researchers for the viable discussion of cosmic problems in a scalar–tensor framework [14–20].

The study of formation and evolution of stars, galaxies, and a cluster of galaxies has important implications in cosmology and gravitational physics. Many observational and experimental surveys such as the Sloan Digital Sky Survey, University of Washington N-Body shop, the Virgo consortium, Leiden observatory, and the Hubble telescope indicate stellar structures to resolve cosmic issues like dark matter, dark energy, and the completeness of big-bang theory. It is conventional that stellar models are mostly rotating and anisotropic in nature. Anisotropy plays a significant role in different dynamical phases of stellar evolutions [21–28].

The patterns of uniform as well as differential rotations of various evolving celestial bodies are investigated through analysis of stability and oscillations of axial configurations in weak approximations. Arutyunyan et al. [29] used the Newtonian (N) and post-Newtonian (pN) regimes to explore the structure of a rotating celestial object. Chandrasekhar and Friedman [30–32] described the perturbation theory of axial symmetric models to discuss the instability ranges of

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uniformly rotating stars. Clifford [33] explored oscillations and stability of a differentially rotating axial symmetric system. Sharif and Bhatti [34] discussed reflection symmetric axial non-static models and found that the instability ranges depend upon the stiffness parameter; and also the spinning models are more stable.

Many researchers [18–20,35–40] investigated stellar evolutions in the modified theories. Since the evolution of such models passes through different dynamical stages, this study can lead to the correct theory of gravity or it may reveal some modifications hidden in the structure formation of the universe. In this paper, we explore the dynamics of non-static axial reflection model in the framework of SBD gravity and study stellar evolution under Mach's principle. The paper is organized in the following format. In the next section, we review SBD gravity and the axial system with reflection symmetry as well as an anisotropic fluid. Section 3 describes the dynamical picture of evolving axial systems, such as hydrodynamics, oscillations, and the instability regimes. The final section summarizes the results.

2 Self-interacting Brans–Dicke theory

The SBD theory is represented by the following action [11–13]:

$$S=\int d^4x\sqrt{-g}\left[\phi R-\frac{\omega_{\rm BD}}{\phi}\nabla^{\mu}\phi\nabla_{\mu}\phi-V(\phi)+L_{\rm m}\right], \tag{1}$$

where $L_{\rm m}$ is the matter distribution and $8\pi G_0=c=1$. Varying the above action with respect to $g_{\mu\nu}$ and ϕ, we obtain

$$G_{\mu\nu}=\frac{1}{\phi}(T^{\rm m}_{\mu\nu}+T^{\phi}_{\mu\nu}), \tag{2}$$

$$\Box\phi=\frac{T^{\rm m}}{3+2\omega_{\rm BD}}+\frac{1}{3+2\omega_{\rm BD}}\left[\phi\frac{dV(\phi)}{d\phi}-2V(\phi)\right]. \tag{3}$$

Here $G_{\mu\nu}$ represents the Einstein tensor, $\frac{T^{\rm m}_{\mu\nu}}{\phi}$ indicates the contribution of matter in the presence of a scalar field, $T^{\rm m}=g^{\mu\nu}T_{\mu\nu}$, and $\Box$ is the d'Alembertian operator. The energy contribution due to the scalar field is described by

$$T^{\phi}_{\mu\nu}=\phi_{,\mu;\nu}-g_{\mu\nu}\Box\phi+\frac{\omega_{\rm BD}}{\phi}\left[\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right]-\frac{V(\phi)}{2}g_{\mu\nu}, \tag{4}$$

which is the energy-momentum tensor, associated with Machian terms, that describes the interaction of the scalar field with the geometry of the distant matter distributions in the universe and the effects of its potentials upon them.

Equations (2) and (3) represent the SBD field equations and the SBD wave equation, respectively. The right hand side of Eq. (2) indicates that both terms are sources of geometry (gravitation). There also exists a static field in the axial symmetric SBD model which has $\phi=\phi(t_0)=$ constant with respect to cosmic time t_0 and generalizes the Einstein equations with an effective cosmological constant $V(\phi_0)$ [41–44]. These static field configurations lead to the dynamics of the non-static axial system.

In order to discuss the dynamics of non-static axial symmetric configurations, we consider a non-static axially symmetric spacetime characterized by reflection [34,45,46],

$$\begin{aligned}ds^2&=-A^2(t,r,\theta)dt^2+B^2(t,r,\theta)(dr^2+r^2d\theta^2)\\&\quad+2L(t,r,\theta)dtd\theta+C^2(t,r,\theta)d\phi^2,\end{aligned} \tag{5}$$

having a matter contribution in the form of a locally anisotropic fluid given by

$$T^{m}_{\mu\nu}=(\rho+p)u_{\mu}u_{\nu}+pg_{\mu\nu}+\Pi_{\mu\nu}. \tag{6}$$

Here

$$\begin{aligned}\Pi_{\mu\nu}&=\frac{1}{3}(\Pi_{II}+2\Pi_{I})\left(k_{\mu}k_{\nu}-\frac{1}{3}h_{\mu\nu}\right)\\&\quad+\frac{1}{3}(\Pi_{I}+2\Pi_{II})\left(\chi_{\mu}\chi_{\nu}-\frac{1}{3}h_{\mu\nu}\right)\\&\quad+\Pi_{k\chi}(k_{\mu}\chi_{\nu}+k_{\nu}\chi_{\mu}),\end{aligned}$$

with

$$\begin{aligned}h_{\mu\nu}&=g_{\mu\nu}+u_{\mu}u_{\nu},\quad \Pi_{k\chi}=k^{\mu}\chi^{\nu}T_{\mu\nu},\quad p=\frac{1}{3}h^{\mu\nu}T_{\mu\nu},\\ \Pi_{I}&=(2k^{\mu}k^{\nu}-s^{\mu}s^{\nu}-\chi^{\mu}\chi^{\nu})T_{\mu\nu},\\ \Pi_{II}&=(2\chi^{\mu}\chi^{\nu}-k^{\mu}k^{\nu}-s^{\mu}s^{\nu})T_{\mu\nu},\end{aligned}$$

where ρ is the energy density, p is for the isotropic pressure, $\Pi_{\mu\nu}$ represents the anisotropic stress tensor, $\Pi_I\neq\Pi_{II}\neq\Pi_{k\chi}$ are the anisotropic scalars, and $h_{\mu\nu}$ expresses a projection tensor. The four-velocity u_{μ}, the unit four-vectors $k_{\mu},\ s_{\mu}$, and χ_{μ} are calculated as

$$\begin{aligned}u_{\mu}&=-A\delta^0_{\mu}+\frac{L}{A}\delta^2_{\mu},\quad k_{\mu}=B\delta^2_{\mu},\quad s_{\mu}=C\delta^3_{\mu},\\ \chi_{\mu}&=\frac{(\Delta)^{1/2}}{A}\delta^2_{\mu},\end{aligned} \tag{7}$$

with $\Delta=r^2A^2B^2+L^2$, and they satisfy the following relations:

$$\begin{aligned}-u^{\mu}u_{\mu}&=s^{\mu}s_{\mu}=k^{\mu}k_{\mu}=\chi^{\mu}\chi_{\mu}=1,\\ s^{\mu}u_{\mu}&=k^{\mu}u_{\mu}=\chi^{\mu}u_{\mu}=s^{\mu}k_{\mu}=\chi^{\mu}k_{\mu}=s^{\mu}\chi_{\mu}=0.\end{aligned} \tag{8}$$

The non-zero components of the energy-momentum tensor due to the scalar field can be represented as

$$\frac{T^{\mu\nu(\phi)}}{\phi}=\begin{pmatrix} v_1+w_1 & x_1+y_1 & x_3+y_3 & 0\\ x_1+y_1 & v_2+w_2 & x_2+y_2 & 0\\ x_3+y_3 & x_2+y_2 & v_3+w_3 & 0\\ 0 & 0 & 0 & v_4+w_4\end{pmatrix}. \qquad (9)$$

Here v_i+w_i represents the diagonal and x_j+y_j shows the non-diagonal components of the stress tensor (4) in which w_i and y_j indicate axial reflection effects due to the scalar field.

3 Dynamics

In this section, we carry out a dynamical analysis of an axial reflection symmetric system. For this purpose, we derive the hydrodynamical equations and discuss oscillations as well as the instability ranges of the perturbed axial system.

3.1 Hydrodynamics

The dynamical equations representing the hydrodynamics of the axially symmetric system can be obtained with the help of the Bianchi identity $G^{\mu\nu}_{;\nu}=0$. This identity along with Eqs. (2), (4) and (6) provides the following equations for $\mu=0,1,2$:

$$\begin{aligned}&\dot{\rho}_{(m\phi)}-\rho_{(m\phi)}\left[\frac{\dot{B}}{B}+\frac{\dot{C}}{C}+\frac{r^2A\dot{A}B^2}{\Delta}+\frac{L\dot{L}}{\Delta}+\frac{r^2A^2B\dot{B}}{\Delta}\right]\\&\quad+(\rho_{(m\phi)}+p_{(m\phi)})\\&\quad\times\frac{AB^2}{\Delta}\left[\frac{2r^2\dot{B}}{B}+\frac{r^2\dot{C}}{C}+\frac{L^2\dot{B}}{A^2B^3}-\frac{L^2\dot{A}}{A^3B^2}\right.\\&\quad\left.+\frac{L\dot{L}}{A^2B^2}+\frac{L^2\dot{C}}{A^2B^2C}\right]\\&\quad+\frac{\Pi_{I(m\phi)}}{3\Delta}\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)+\frac{\Pi_{II(m\phi)}}{\Delta}\\&\quad\times\left[r^2A^2B^2\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)+\frac{L^2\dot{L}}{AL}-\frac{L^2\dot{A}}{A^2}\right.\\&\quad\left.-\frac{L^2\dot{C}}{AC}\right]+E_0(t,r,\theta)=0, \qquad (10)\end{aligned}$$

$$\begin{aligned}&p'_{(m\phi)}+\frac{2}{9}\left(2\Pi'_{I(m\phi)}+\Pi'_{II(m\phi)}\right)\\&\quad+\left[p_{(m\phi)}+\frac{2}{9}(2\Pi_{I(m\phi)}+\Pi_{II(m\phi)})\right]\\&\quad\times\left[\frac{C'}{C}+\frac{3LL'}{2}+\frac{r^2AA'B^2}{\Delta}+\frac{r^2A^2BB'}{\Delta}\right.\\&\quad\left.+\frac{2rA^2B^2}{\Delta}-\frac{rA^2B(rB)'}{\Delta}\right]\\&\quad-\frac{r^2AB^5}{(\Delta)^{3/2}}\left[\Pi^{\theta}_{k\chi(m\phi)}-\Pi_{k\chi(m\phi)}\right.\\&\quad\times\left(\frac{A^\theta}{A}-\frac{6B^\theta}{B}-\frac{C^\theta}{C}-\frac{4r^2A^2B^2}{\Delta}\left(\frac{A^\theta}{A}+\frac{B^\theta}{B}\right)\right.\\&\quad\left.\left.-\frac{4LL^\theta}{\Delta}\right)\right]+\frac{\rho_{(m\phi)}r^4A^4B^4}{\Delta^2}\left(B\dot{B}+\frac{A'}{A}-\frac{LA^\theta}{r^2AB^2}\right)\\&\quad-\frac{\rho_{(m\phi)}r^2A^2L^2B^2}{\Delta^2}\\&\quad\times\left(\frac{(rB)'}{rB}+\frac{L}{2L'}\right)+E_1(t,r,\theta)=0, \qquad (11)\end{aligned}$$

$$\begin{aligned}&\frac{\rho_{(m\phi)}A^2B^2L}{\Delta^2}\left[\frac{r^2\dot{\rho}_{(m\phi)}}{\rho_{(m\phi)}}+\frac{r^2\dot{A}}{A}+3\frac{r^2\dot{B}}{B}+\frac{r^2\dot{L}}{L}+\frac{r^2\dot{C}}{C}\right.\\&\quad+\frac{1}{B^2}\left(\frac{\rho^\theta_{(m\phi)}}{\rho_{(m\phi)}}+\frac{2L^\theta}{L}+\frac{2A^\theta}{A}\right)\\&\quad+\frac{1}{\Delta}4r^5A^2\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)-\frac{4\dot{L}}{L\Delta}-\frac{5LAA^\theta}{\Delta}\\&\quad-\frac{LA^2B^\theta}{B\Delta}+\frac{r^2A^2B^2}{\Delta}\left(\frac{\dot{L}}{L}+\frac{\dot{B}}{B}\right)\\&\quad\left.+\frac{r^2A^3B^2A^\theta}{L\Delta}-\frac{4L^2L^\theta\Delta}{r^2B^2}\right]\\&\quad+\frac{\rho_{(m\phi)}A^2L^2}{\Delta^2}\left(\frac{B^\theta}{B}+\frac{C^\theta}{C}\right.\\&\quad\left.-\frac{r^2BL\dot{B}}{\Delta}\right)-\frac{r^3AB^3\Pi_{k\chi(m\phi)}}{\Delta^{3/2}}\left[\frac{\Pi'_{k\chi(m\phi)}}{\Pi_{k\chi(m\phi)}}+\frac{3}{r}\right.\\&\quad+\frac{4B'}{B}+\frac{A'}{A}+\frac{C'}{C}+\frac{3LL'}{2\Delta}\\&\quad\left.+\frac{3r^2A^2B^2}{\Delta}\left(\frac{3}{r}+\frac{2A'}{A}+\frac{3B'}{B}\right)+\frac{7LL'}{2\Delta}\right]\\&\quad+\frac{1}{\Delta}\left(p_{(m\phi)}+\frac{2}{9}\left(\Pi_{I(m\phi)}\right.\right.\\&\quad\left.\left.+2\Pi_{II(m\phi)}\right)\right)\frac{1}{\Delta}\left[\frac{r^2A^2B^2}{\Delta}\left\{(2A^2+A)\left(\frac{A^\theta}{A}+\frac{B^\theta}{B}\right)\right.\right.\\&\quad-L\frac{\dot{B}}{B}+\frac{2AB^\theta}{B}+2AA^\theta\\&\quad\left.+\frac{A^2C^\theta}{C}-\frac{r^2BL\dot{B}}{\Delta}-\frac{2A^2LL^\theta}{\Delta}-\frac{L\dot{B}}{B}\right]-\frac{p}{C\Delta}\\&\quad\times(L\dot{C}+A^2C^\theta)+\frac{A^2}{\Delta}\left\{p^\theta+\frac{2}{9}(\Pi^\theta_{I,(m\phi)}+2\Pi^\theta_{II,(m\phi)})\right\}\\&\quad+E_2(t,r,\theta)=0. \qquad (12)\end{aligned}$$

Here a dot, a prime, and the superscript θ indicate derivatives with respect to time, r, and θ, respectively. The subscript $(m\phi)$ implies energy-momentum terms of the matter distribution per scalar field $\left(\frac{T^{\mu\nu(m)}}{\phi}\right)$, which corresponds to contributions of matter dynamics in the presence of the scalar field. The terms $E_0(t,r,\theta)$, $E_1(t,r,\theta)$, and $E_2(t,r,\theta)$ represent energy contributions due to scalar field and their values are given in Eqs. (A2)–(A3). Equations (10)–(12) describe hydrodynamical equations of an axial reflection symmetric fluid in SBD gravity.

3.2 Oscillations

Now we discuss oscillations of the axial system through a perturbation approach. We assume that initially the system is in hydrostatic equilibrium and after that all metric functions along with the dynamical variables are perturbed with a time dependent perturbation $T(t) = e^{iwt}$ and the system starts oscillating with frequency w. The metric tensor as well as the scalar field and scalar potential has the same time dependence, while the dynamical variables bear the same time dependence as follows:

$$A(t,r,\theta) = A_0(r,\theta) + \epsilon e^{iwt} a(r,\theta), \tag{13}$$
$$B(t,r,\theta) = B_0(r,\theta) + \epsilon e^{iwt} b(r,\theta), \tag{14}$$
$$C(t,r,\theta) = C_0(r,\theta) + \epsilon e^{iwt} c(r,\theta), \tag{15}$$
$$L(t,r,\theta) = L_0(r,\theta) + \epsilon e^{iwt} l(r,\theta), \tag{16}$$
$$\phi(t,r,\theta) = \phi_o(r,\theta) + \epsilon e^{iwt} \Phi(r,\theta), \tag{17}$$
$$V(\phi) = V_0(r,\theta) + \epsilon e^{iwt} \bar{V}(r,\theta), \tag{18}$$
$$p(t,r,\theta) = p_0(r,\theta) + \epsilon \bar{p}(iwt,r,\theta), \tag{19}$$
$$\rho(t,r,\theta) = \rho_0(r,\theta) + \epsilon \bar{\rho}(iwt,r,\theta), \tag{20}$$
$$\Pi_I(t,r,\theta) = \Pi_{I0}(r,\theta) + \epsilon \bar{\Pi}_I(iwt,r,\theta), \tag{21}$$
$$\Pi_{II}(t,r,\theta) = \Pi_{II0}(r,\theta) + \epsilon \bar{\Pi}_{II}(iwt,r,\theta), \tag{22}$$
$$\Pi_{k\chi}(t,r,\theta) = \Pi_{k\chi 0}(r,\theta) + \epsilon \bar{\Pi}_{k\chi}(iwt,r,\theta). \tag{23}$$

Here $0 < \epsilon \ll 1$ and the subscript zero indicates a static distribution, while terms having a bar represent perturbed terms [34,47].

Using Eqs. (13)–(23), the perturbed configuration of 02-components of the field equations (2) can be represented as

$$(lw^2 + mw + n)e^{iwt} = 0,$$

where values of l, m and n are given in (A4)–(A6). Since $e^{iwt} \neq 0$, this implies that

$$w = \frac{-m + \sqrt{m^2 - 4ln}}{2l}, \tag{24}$$

yielding the frequency of the oscillating axial reflection system. This shows that the frequency of the oscillations depends upon the DE source (scalar field), anisotropic effects, and the reflection configuration.

The total density of the oscillating system can be obtained from the perturbed form of the first law of conservation (10) as follows:

$$\bar{\rho}_{(m\phi)} = [(F_{(m\phi)} + \bar{E}_{0(a)})iw + \bar{E}_{0(b)}]e^{iwt}. \tag{25}$$

Here $F_{(m\phi)}$ shows the contribution of matter with a scalar field, and $\bar{E}_{0(a)}$ and $\bar{E}_{0(b)}$ represent the scalar field distributions whose values are given in Eqs. (A7)–(A8). The terms with subscript (a) represent a scalar field coupled to the frequency, while subscript (b) shows a scalar field without frequency. The perturbed form of Eq. (11) provides the equation of motion of the oscillating system, given by

$$\frac{1}{B_0^2}\left\{\bar{p}'_{(m\phi)} + \frac{2}{9}(2\bar{\Pi}'_{I(m\phi)} + \bar{\Pi}'_{II(m\phi)})\right\}$$
$$+ \frac{1}{B_0^2}\left\{\bar{p}_{(m\phi)} + \frac{2}{9}(2\bar{\Pi}_{I(m\phi)} + \bar{\Pi}_{II(m\phi)})\right\}$$
$$\times\left\{\frac{C'_0}{C_0} + \frac{3L_0L'_0}{2} + \frac{r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A'_0}{A_0} + \frac{B'_0}{B_0} + \frac{2}{r} - \frac{1}{r} - \frac{B'_0}{B_0}\right)\right\}$$
$$- \frac{r^2A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\bar{\Pi}^{\theta}_{k\chi(m\phi)}$$
$$- \bar{\Pi}_{k\chi(m\phi)}\frac{r^2A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\left\{\frac{A_0^{\theta}}{A_0} + \frac{6B_0^{\theta}}{B_0} + \frac{C_0^{\theta}}{C_0} + \frac{4L_0L_0^{\theta}}{\Delta_0}\right.$$
$$\left. + \frac{4r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_0^{\theta}}{A_0} + \frac{B_0^{\theta}}{B_0}\right)\right\}$$

$$+ \frac{\bar{\rho}_{(m\phi)}r^4A_0^4}{\Delta_0^2}\left(\frac{A'_0}{A_0} - \frac{L_0A_0^{\theta}}{r^2A_0B_0^2}\right)$$
$$- \bar{\rho}_{(m\phi)}L_0^2r^2\frac{A_0^2}{\Delta_0^2}\left\{\frac{L_0}{2} + \frac{1}{r} + \frac{B'_0}{B_0}\right\} - e^{iwt}\left[\frac{2b}{B_0^3}\right.$$
$$\times\left\{p'_{0(m\phi)} + \frac{2}{9}(2\Pi'_{I0(m\phi)} + \Pi'_{II0(m\phi)})\right\} - \left[\left\{\left(\frac{c}{C_0}\right)'\right.\right.$$
$$\left.\left. + \frac{3L_0L'_0}{2}\left(\frac{l}{L_0} + \frac{l'}{L'_0}\right)\right\}\right.$$
$$+ \frac{r^2A_0^2B_0^2}{\Delta_0^2}\left(\frac{2a}{A_0} + \frac{2b}{B_0} - \frac{\Delta_p}{\Delta_0}\right)\left(\frac{A'_0}{A_0} + \frac{B'_0}{B_0} + \frac{2}{r} - \frac{1}{r} - \frac{B'_0}{B_0}\right)$$
$$+ \frac{r^2A_0^2B_0^2}{\Delta_0}$$
$$\left.\times\left\{\left(\frac{a}{A_0} + \frac{2b}{B_0}\right)' - \left(\frac{b}{B_0}\right)'\right\}\right]\left\{p_{0(m\phi)}\right.$$
$$\left. + \frac{2}{9}(2\Pi_{I0(m\phi)} + \Pi_{II0(m\phi)})\right\}\frac{1}{B_0^2} - \frac{2b}{B_0^2}$$
$$\times\left\{p_{0(m\phi)} + \frac{2}{9}(2\Pi_{I0(m\phi)} + \Pi_{II0(m\phi)})\right\}\left\{\frac{C'_0}{C_0} + \frac{3L_0L'_0}{2}\right.$$
$$\left. + \frac{r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A'_0}{A_0} + \frac{B'_0}{B_0} + \frac{2}{r} - \frac{1}{r} - \frac{B'_0}{B_0}\right)\right\}$$
$$+ \frac{r^2A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\Pi^{\theta}_{k\chi 0(m\phi)}\left(\frac{a}{A_0} + \frac{3b}{B_0} - \frac{3\Delta_p}{\Delta_0}\right)$$

$$
\begin{aligned}
&+\frac{r^3A_0B_0^3\Pi_{k\chi0(m\phi)}}{\Delta_0^{\frac{3}{2}}}\\
&\times\left(\frac{a}{A_0}+\frac{3b}{B_0}-\frac{3\Delta_p}{\Delta_0}\right)\left\{\frac{A_0^\theta}{A_0}+\frac{6B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}+\frac{4L_0L_0^\theta}{\Delta_0}\right.\\
&\left.+\frac{4r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_0^\theta}{A_0}+\frac{B_0^\theta}{B_0}\right)\right\}\\
&+\frac{r^3A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\Pi_{k\chi0(m\phi)}\left[\frac{6B_0^\theta}{B_0}\left(\frac{b^\theta}{B_0^\theta}+\frac{b}{B_0}\right)\left(\frac{a}{A_0}+\frac{c}{C_0}\right)^\theta\right.\\
&+\frac{4L_0L_0^\theta}{\Delta_0}\left(\frac{l}{L_0}+\frac{l^\theta}{L_0^\theta}-\frac{\Delta_p}{\Delta_0}\right)\\
&+\frac{4r^2A_0^2B_0^2}{\Delta_0}\left(\frac{2a}{A_0}+\frac{2b}{B_0}-\frac{\Delta_p}{\Delta_0}\right)\\
&\left.\times\left(\frac{a}{A_0}+\frac{b}{B_0}\right)^\theta\right]-\frac{\rho_{0(m\phi)}r^4A_0^4}{\Delta_0^2}\\
&\times\left\{\left(\frac{a}{A_0}\right)'-\frac{L_0}{r^2}\frac{A_0^\theta}{A_0B_0^2}\left(\frac{l}{L_0}+\frac{a^\theta}{A_0^\theta}-\frac{a}{A_0}-\frac{2b}{B_0}\right)\right\}\\
&+\frac{\rho_{0(m\phi)}L_0^2A_0^2r^2}{\Delta_0^2}\left(\frac{2l}{L_0}+\frac{2a}{A_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{\rho_{0(m\phi)}L_0^2A_0^2r^2}{\Delta_0^2}\left\{\frac{l}{2}+\left(\frac{b}{B_0}\right)'\right\}\Bigg]+\bar{E}_1=0. \qquad (26)
\end{aligned}
$$

The superscript p indicates the perturbed form, $\bar{E}_1$ represents perturbed configurations of the scalar field terms given in Eq. (A10).

3.3 Spin-independent oscillations

The local spinning of anisotropic system is calculated through the vorticity tensor. For the axial symmetric spacetime with reflection, the vorticity tensor can be expressed in terms of k_μ and χ_μ as

$$\Omega_{\mu\nu}=\Omega(k_\mu\chi_\nu-\chi_\mu k_\nu),$$

where

$$\Omega=\frac{1}{2B\sqrt{\Delta}}\left(L'-2\frac{A'L}{A}\right)$$

is the vorticity scalar. This shows that the spin-independent motion occurs whenever $\Omega=0$, which is possible if $(L'-2\frac{A'L}{A})=0$. This leads to

$$\ln\left(\frac{L\tilde{K}}{A^2}\right)=0,$$

where $\tilde{K}=\tilde{K}(t,\theta)$ is an arbitrary function of integration. This implies that $L\tilde{K}=A^2$, consequently, if $L=0$ we have $A=0$ and the existence of non-static configuration of axial spacetime is disturbed. Therefore, for $\Omega=0$, we take

$$L\tilde{K}=A^2,\quad L\neq0.$$

Thus, oscillations convert into spin-independent form whenever $L\tilde{K}=A^2$ and $L\neq0$. This implies that spin-independent oscillations depend upon the reflection contribution. Equations (24)–(26) along with $L\tilde{K}=A^2$, $L\neq0$ provide the frequency, total density, and the equation of motion of spin-independent oscillations of the axisymmetric distribution.

3.4 Stability analysis

Here, we discuss the stability of the oscillating collapsing axial system (with reflection symmetry) in the presence of the scalar field. We assume that the system is perturbed adiabatically and satisfies the equation of state [48]

$$\bar{p}=\Gamma\frac{p_0}{\rho_0+p_0}\bar{\rho}, \qquad (27)$$

where the equation of state parameter (Γ) represents a constant adiabatic index which calculates stiffness or rigidity in the fluid. This equation with Eq. (25) provides the perturbed part of anisotropic stresses as follows:

$$
\begin{aligned}
\bar{p}_{(m\phi)}&=-\Gamma\frac{p_{0(m\phi)}}{\rho_{0(m\phi)}+p_{0(m\phi)}}((F_{(m\phi)}+\bar{E}_{0(a)})iw+\bar{E}_{0(b)})e^{iwt},\\
\bar{\Pi}_{I(m\phi)}&=-\Gamma\frac{\Pi_{I0(m\phi)}}{\rho_{0(m\phi)}+\Pi_{I0(m\phi)}}((F_{(m\phi)}+\bar{E}_{0(a)})iw+\bar{E}_{0(b)})e^{iwt},\\
\bar{\Pi}_{II(m\phi)}&=-\Gamma\frac{\Pi_{II0(m\phi)}}{\rho_{0(m\phi)}+\Pi_{II0(m\phi)}}((F_{(m\phi)}+\bar{E}_{0(a)})iw+\bar{E}_{0(b)})e^{iwt},\\
\bar{\Pi}_{k\chi(m\phi)}&=-\Gamma\frac{\Pi_{k\chi0(m\phi)}}{\rho_{0(m\phi)}+\Pi_{k\chi0(m\phi)}}((F_{(m\phi)}+\bar{E}_{0(a)})iw+\bar{E}_{0(b)})e^{iwt}.
\end{aligned}
$$

Using these relations in the equation of motion (26), we obtain the collapse equation (hydrostatic equation) of the oscillating axial reflection system,

$$-\Gamma(\delta_{(m(\mathrm{BD}))}(iw,r,\theta))=-(\lambda_{(m(\mathrm{BD}))}(iw,r,\theta)+\bar{E}_1). \quad (28)$$

The quantity $\delta_{(m(\mathrm{BD}))}$ shows the pressure gradient forces and anti-gravitational forces (due to matter as well as scalar field distributions) coupled to the adiabatic index whereas $\lambda_{(m(\mathrm{BD}))}+\bar{E}_1$ gives gravitational forces (forces opposite to pressure gradients forces) mediated by matter as well as scalar field contributions. The values of these terms are given in (A11) and (A12). The adiabatic index Γ is taken to be positive in order to balance the hydrostatic configurations between pressure gradient as well as gravitational force.

3.5 Newtonian approximations

In order to evaluate stability criteria in the N limits, we use the approximation $A_0=1,\ B_0=1,\ C_0=r,\ L_0=r,\ \Delta_0=r^2,\ \phi=\phi_0$, and $V(\phi)=V_0$. By using these limits in Eq. (28), it follows that

$$-\Gamma\left(\delta_{(m(\mathrm{BD}))N}(iw,r,\theta)\right)=-\left(\lambda_{(m(\mathrm{BD}))N}(iw,r,\theta)+\bar{E}_{1(N)}\right), \tag{29}$$

where the values of $\delta_{(m(\mathrm{BD}))N}$, $\lambda_{(m(\mathrm{BD}))N}$, and $\bar{E}_{1(N)}$ are given in (A13)–(A15). This provides a hydrostatic condition which implies that the system collapses whenever

$$-\Gamma(\delta_{(m(\mathrm{BD}))N})<-(\lambda_{(m(\mathrm{BD}))N}+\bar{E}_{1(N)}),$$

or

$$\Gamma<\frac{-(\lambda_{(m(\mathrm{BD}))N}+\bar{E}_{1(N)})}{-\delta_{(m(\mathrm{BD}))N}}. \tag{30}$$

For $\Gamma>0$, we need to take $|(\lambda_{(m(\mathrm{BD}))N}+\bar{E}_{1(N)})|$ and $|\delta_{(m(\mathrm{BD}))N}|$. Thus the system remains unstable (collapses) as long as the inequality (30) holds. This implies that the instability ranges in the N limits can be calculated through the stiffness of the fluid (adiabatic index), which depends upon the configurations of pressure gradient forces as well as anti-gravitational forces coupled to the adiabatic index and gravitational forces. These factors in turn depend upon the energy density, anisotropies, reflection effects, and scalar field contributions. We can summarize the results as follows:

- If the gravitational forces $|(\lambda_{(m(\mathrm{BD}))N}+\bar{E}_{1(N)})|$ are balanced by anti-gravitational and pressure gradient forces $(\delta_{(m())N})$, then (29) implies that $\Gamma=1$ and the system is in complete hydrostatic equilibrium (remains stable).
- If the anti-gravitational and pressure gradient distribution related to stiffness parameter are greater than gravitational contribution, then according to (29), the system becomes unstable (but not collapses) for $0<\Gamma<1$.
- Equation (29) implies that if gravitational effects are greater than that of the anti-gravitational and pressure gradient effects coupled to the adiabatic index, the system collapses, leading to instability for $\Gamma>1$.

In the case of spin-independent oscillations, the inequality (30) with $L\tilde{K}=A^2$ and $L\neq 0$ provides the criteria for unstable spin-independent oscillations. The numerical instability ranges ($0<\Gamma<1$ and $\Gamma>1$) remain the same as calculated for spin-dependent oscillations.

3.6 Post-Newtonian approximation

In pN limits, we use approximations up to the order of $\frac{m_0}{r_1}$ (discarding terms having higher order of $\frac{m_0}{r_1}$) as follows: $A_0=1-\frac{m_0}{r_1}$, $B_0=1-\frac{m_0}{r_1}$, $\phi=\phi_0+\varphi$, and $V=V_0+\varphi V_0'$ [49], φ represents local deviations of the scalar field from ϕ_0. The axial system becomes unstable in the pN limits if the adiabatic index satisfies the following inequality:

$$\Gamma<\frac{(\lambda_{(m(\mathrm{BD}))\mathrm{pN}}+\bar{E}_{1(\mathrm{pN})})}{\delta_{(m(\mathrm{BD}))\mathrm{pN}}},$$

the values of $\delta_{(m(\mathrm{BD}))\mathrm{pN}}$ and $(\lambda_{(m(\mathrm{BD}))\mathrm{pN}}+\bar{E}_{1(\mathrm{pN})})$ are given in (A16)–(A18). Similar to the N case, the instability criteria depend upon the rigidity of the fluid and $\Gamma=1$ provides stable configurations in the pN regime, while the system becomes unstable for other values of the adiabatic index.

4 Concluding remarks

According to recent observation, DE controls the dynamics of the expansion of the present universe. General relativity is considered as a fundamental theory for the description of various astrophysical processes. It is an excellent theory of gravity which has many achievements but is said to break down at the Planck length. Its proposed DE candidate, the "cosmological constant" is not considered to be compatible with the calculated vacuum energy of quantum fields. This non-normalizable behavior of general relativity induces the concept of alternative theories of gravity (alternative candidates of DE) [50]. These theories are constructed by incorporating extra degrees of freedom in the Einstein–Hilbert Lagrangian density either in the geometrical (gravitational) or the matter part. Some of these DE models are the Chaplygin gas, tachyon fields, quintessence, k-essence, and modified gravities such as $f(R)$ gravity, $f(T)$ theory, Gauss–Bonnet gravity, $f(R,T)$ gravity, and scalar–tensor theories.

The scalar–tensor theory of gravity is an alternative step to unify theoretically gravity and quantum mechanics at high energies by introducing a scalar field as an extra degree of freedom in the Einstein–Hilbert Lagrangian density. Brans–Dicke gravity is the first proper scalar–tensor gravity, seen as a prototype of an alternative theory for Einstein gravity. The principal features of this theory are the compatibility with Dirac's hypothesis and Mach's principle, i.e., this theory corresponds to a dynamical gravitational coupling (dynamical gravitational constant) by means of a dynamical scalar field ϕ (extra force field) which allows the distributions of distant matter to affect the dynamics at a point. The basic idea of this theory is that the inertial mass of an object is not an intrinsic property of the object itself but is generated by the gravitational effect of all the other matter in the universe. In this way, this theory has generalized the Einstein gravity to a Machian one (compatible with Mach's principle) and has provided convenient solutions of many cosmic problems, especially the accelerated expansion of the universe.

In general relativity, the effects of stellar rotations cannot be neglected in a full investigation of the formation of stars and black holes. During evolution, the self-gravitating fluid passes through many phases of dynamical activities that remain in hydrostatic equilibrium for a short span. In this paper, we have studied the dynamical stability of a non-static stellar model with an axial reflection symmetric anisotropic fluid distribution under the influence of dynamical gravitational coupling through SBD gravity. We have generalized the dynamical analysis of general relativity by incorporating Mach's principle to explore the effects of DE upon the cosmic evolution

When a system has departed from its initial static phase, it becomes perturbed and starts oscillating. We have explored oscillations of the axial reflection configuration under a time-frequency dependent perturbation. It is shown that the frequency of the rotating oscillations depends upon anisotropic effects, the reflection symmetry, and the DE contribution represented by the scalar field. The perturbed form of the conservation laws yields the total energy density and equation of motion, which depend upon the behavior of anisotropic effects as well as frequency. We have also investigated the spin-independent oscillation and found that reflection configuration is the factor which controls the spin of the axial system.

In order to obtain viable models of the rotating system, we have studied various instability ranges with the help of a collapse equation. It is found that the stable configurations of the spin-dependent oscillations depend upon the stiffness of the fluid, which in turn depends upon anisotropic effects, and the reflection parameter as well as the distribution of the scalar field. The instability of the spin-independent oscillating system depends upon the rigidity of the fluids due to anisotropy, and reflection effects with the constraints $L\tilde{K}=A^2, L\neq 0$ as well as an SBD gravity contribution. We would like to mention here that the dynamics of the axial rotating system in general relativity depends upon the anisotropic as well as reflection effects, but here the results are modified by the inclusion of an extra field (scalar field) as a DE candidate. Thus we can conclude that in the present accelerating universe, DE not only controls the expansion among celestial objects but it also affects stellar evolution.

According to Mach's principle, the dynamics of any evolving body in the universe is not an intrinsic property, but the surrounding distant matter also has its effect. In this way, it involves all the surrounding stellar structures in the analysis. It would be interesting to explore the collapse phenomenon and its consequences on the stellar objects according to Mach's principle to contribute to the study of stellar evolution in the presence of DE.

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Appendix A

The values of the scalar field energy terms $E_0(t,r,\theta)$, $E_1(t,r,\theta)$, and $E_2(t,r,\theta)$ are given by

$$\begin{aligned}
E_0(t,r,\theta) &= \dot{v}_1+\dot{w}_1+x_1'+y_1'+x_3^\theta+y_3^\theta\\
&+\left(\frac{2B^2r^2A\dot{A}}{\Delta}+2\frac{L\dot{L}}{\Delta}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right.\\
&\left.+\frac{LAA^\theta}{A}+\frac{ABr^2\dot{B}}{\Delta}\right)(v_1+w_1)+\left(3\frac{B^2AA'r^2}{\Delta}\right.\\
&+2\frac{A^2Br^2B'}{\Delta}+\frac{2LL'}{\Delta}+\frac{B'}{B}\\
&\left.+\frac{C'}{C}+\frac{A^2B^2r}{\Delta}\right)(x_1+y_1)+\left(\frac{3B^2AA^\theta r^2}{\Delta}+\frac{3B\dot{B}r^2}{\Delta}\right.\\
&+\frac{B^\theta}{B}+\frac{LL^\theta}{\Delta}\\
&\left.+\frac{C^\theta}{C}-\frac{LB\dot{B}r^2}{\Delta}+\frac{A^2BB^\theta r^2}{\Delta}\right)(x_3+y_3)\\
&+\left(\frac{B^3r^2\dot{B}}{\Delta}-\frac{BLB^\theta}{\Delta}\right)(v_2+w_2)\\
&+\left(\frac{B^2r^2L^\theta}{\Delta}-\frac{B^2r^2\dot{B}}{\Delta}-\frac{LB^\theta}{\Delta}\right)(v_3+w_3)\\
&+\left(\frac{CB^2r^2\dot{C}}{\Delta}-\frac{LC^\theta}{\Delta}\right) \qquad\qquad (A1)\\
&\times(v_4+w_4)+\left(\frac{B^2r^2L'}{\Delta}+\frac{LrB'}{\Delta}+\frac{LB}{\Delta}\right)(x_2+y_2),
\end{aligned}$$

$$\begin{aligned}
E_1(t,r,\theta) &= (\dot{x}_1+\dot{y}_1)+(v_2+w_2)'+(v_1+w_1)^\theta\\
&+(v_1+w_1)\left(\frac{AA'}{B^2}\right)\\
&+\left(\frac{B^2r^2AA'}{\Delta}+\frac{3LL'}{2\Delta}+2\frac{B'}{B}+\frac{C'}{C}+\frac{A^2BB'r^2}{\Delta}\right.\\
&\left.+\frac{A^2B^2r}{\Delta}\right)(v_2+w_2)\\
&-(v_3+w_3)\frac{r(rB'-B)}{B}+\left(\frac{3\dot{B}}{B}+\frac{B^2r^2AA^\theta}{\Delta}+\frac{L\dot{L}}{\Delta}\right.\\
&\left.-\frac{LAA^\theta}{\Delta}+\frac{AB\dot{B}r^2}{\Delta}\right)\\
&\times(x_1+y_1)+\left(3\frac{B^\theta}{B}+\frac{B^2Ar^2A^\theta}{\Delta}+\frac{L\dot{B}}{\Delta}\right.\\
&\left.+\frac{LL^\theta}{\Delta}-\frac{LB\dot{B}r^2}{\Delta}+\frac{A^2BB^\theta r^2}{\Delta}\right) \qquad\qquad (A2)\\
&\times(x_2+y_2)-\frac{CC'}{B^2}(v_4+w_4)-\frac{L'}{B^2}(x_3+y_3),
\end{aligned}$$

$$\begin{aligned}
E_2(t,r,\theta) &= (\dot{x}_3+\dot{y}_3)+(x_2+y_2)'+(v_3+w_3)^\theta\\
&+\left(\frac{AL'}{2\Delta}-\frac{ALA'}{\Delta}\right)\\
&\times(x_1+y_1)+\left(\frac{3AB\dot{B}r^2}{\Delta}+\frac{B^2r^2A\dot{A}}{\Delta}+\frac{L\dot{L}}{\Delta}\right.\\
&\left.+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}-\frac{2ALA^\theta}{\Delta}\right)\\
&\times(x_3+y_3)+\left(2\frac{LL'}{\Delta}+\frac{3A^2BB'r^2}{2\Delta}+\frac{3A^2B^2r}{\Delta}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{B'}{B}+\frac{C'}{C}\Bigg)(x_2+y_2)\\
&+\Bigg(\frac{A^2BB^{\theta}r^2}{\Delta}-\frac{3LL^{\theta}}{\Delta}-\frac{B^2r^2AA^{\theta}}{\Delta}\\
&+\frac{Br^2L\dot{B}}{\Delta}+\frac{B^{\theta}}{B}+\frac{C^{\theta}}{C}\Bigg)(v_3+w_3)\\
&-\Bigg(\frac{CL\dot{C}}{\Delta}+\frac{A^2C^{\theta}}{\Delta}\Bigg)(v_4+w_4). \qquad \text{(A3)}
\end{aligned}
$$

In the perturbed configuration of 02-component of Eq. (2), the resulting values of $l,\ m$, and n are

$$
\begin{aligned}
l=-&\left[-\frac{r^2cB_0^2L_0^3}{C_0\Delta_0^2}-\frac{r^4cA_0^2B_0^4L_0}{C_0\Delta_0^2}-\frac{L_0^3B_0r^2b}{\Delta_0^2}\right]\\
&(x'_{3(p1)}+y'_{3(p1)}), \qquad \text{(A4)}
\end{aligned}
$$

$$
\begin{aligned}
m=i&\Bigg[-r^2aL_0^2A_0^2B_0^2+\frac{L_0B_0A_0^2r^2B_0^{\theta}l}{\Delta_0^2}\\
&+\frac{C^{\theta}L_0^4}{C_0\Delta_0^2}+\frac{4L_0B_0^4r^2C_0^{\theta}l}{C_0\Delta_0^2}\\
&+\frac{L_0A_0^2B_0^2r^2L_0^{\theta}c}{C_0A_0^2}-\frac{r^2B_0^2aA_0C_0^{\theta}}{C_0\Delta_0^2}\\
&+\frac{L_0A_0^2B_0r^2L_0^{\theta}b}{\Delta_0^2}-\frac{B_0^3b^{\theta}A_0^2r^4}{\Delta_0^2}\\
&+\frac{L_0^5C_0^{\theta}b}{B_0C_0\Delta_0^2}+\frac{L_0^5B_0^{\theta}c}{B_0C_0\Delta_0^2}\\
&+\frac{b^{\theta}L_0^4}{B_0\Delta_0^2}-\frac{r^4A_0^4B_0^5c^{\theta}}{\Delta_0^2}\\
&+\frac{br^4A_0^2B_0^2B_0^{\theta}}{\Delta_0^2}+\frac{br^4A_0^3B_0^3A_0^{\theta}}{\Delta_0^2}\\
&+\frac{cr^4B_0^4A_0^3A_0^{\theta}}{C_0\Delta_0^2}+r^4bA_0^4B_0^3C_0^{\theta}C_0\Delta^2\Bigg](x_{3(p2)}+y_{3(p2)}), \qquad \text{(A5)}
\end{aligned}
$$

$$
\begin{aligned}
n=-&\frac{2r^4L_0A_0^3B_0A'_0B'_0}{\Delta_0^4}\left(\frac{l}{L_0}+\frac{3a}{A_0}+\frac{b}{B_0}+\frac{a'}{A_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&-\frac{3}{2}\frac{r^2A_0^2L_0^2L'_0B'_0}{B_0\Delta_0^2}\\
&\times\left(\frac{l'}{L'_0}+\frac{b'}{B'_0}+\frac{2a}{A_0}+\frac{2l}{L_0}-\frac{2\Delta_p}{\Delta_0}-\frac{b}{B_0}\right)\Delta_0^2\\
&-\frac{br^2A_0^2L_0^2B_0^{\theta}}{\Delta_0^2}-\frac{3r^2A_0A'_0L'_0L_0^2}{\Delta_0^2}\\
&\times\left(\frac{a}{A_0}+\frac{a'}{A'_0}+\frac{l'}{L'_0}+\frac{2l}{L_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{3r^2L_0L_0'^2A_0^2}{\Delta_0^2}\left(\frac{l}{L_0}+\frac{2l'}{L'_0}+\frac{2a}{A_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{3rA_0^2B'_0L_0^3}{B_0\Delta_0^2}\left(\frac{2a}{A_0}+\frac{b'}{B'_0}-\frac{b}{B_0}+3l\Delta_pL_0\Delta_0\right)\\
&-\frac{A_0^2L_0^2B_0^{\theta}L_0^{\theta}T}{B-0\Delta_0^2}\Bigg(\frac{2a}{A_0}+\frac{2l}{L_0}+\frac{b^{\theta}}{B_0^{\theta}}\\
&+\frac{l^{\theta}}{L_0^{\theta}}-\frac{b^2\Delta_p}{B_0\Delta_0}\Bigg)-\frac{r^2B'_0C'_0A_0^2L_0^3}{B_0C_0\Delta_0^2}\Bigg(\frac{b'}{B'_0}+\frac{c'}{C'_0}+\frac{2a}{A_0}\\
&+\frac{3l}{L_0}-\frac{b}{B_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Bigg)\\
&-\frac{L_0A_0^4r^4B_0'^2}{\Delta_0^4}\left(\frac{l}{L_0}+\frac{4a}{A_0}+\frac{2b}{B_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{r^4L_0B_0^2A_0^3A'_0C'_0}{C_0\Delta_0^2}\Bigg(\frac{l}{L_0}+\frac{2b}{B_0}+\frac{3a}{A_0}\\
&+\frac{a'}{A'_0}+\frac{c'}{C'_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Bigg)+\frac{r^2A_0A'_0C'_0L_0^3}{C_0\Delta_0^2}\\
&\times\left(\frac{3l}{L_0}+\frac{a'}{A'_0}+\frac{a}{A_0}+\frac{c'}{C'_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{r^2A_0A'_0C'_0L_0^3}{C_0\Delta_0^2}\left(\frac{2a'}{A'_0}+\frac{3l}{L_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{A_0^2B_0^{\theta}C_0^{\theta}L_0^3}{B_0C_0\Delta_0^2}\Bigg(\frac{2a}{A_0}+\frac{b^{\theta}}{B_0^{\theta}}\\
&+\frac{c^{\theta}}{C_0^{\theta}}-\frac{b}{B_0}-\frac{c}{C_0}+\frac{3l}{L_0}-\frac{2\Delta_p}{\Delta_0}\Bigg)-\frac{4A_0^2L_0^2L_0^{\theta}C_0^{\theta}}{C_0\Delta_0^2}\\
&\times\Bigg(\frac{2a}{A_0}+\frac{2l}{L_0}+\frac{l^{\theta}}{L_0^{\theta}}+\frac{c^{\theta}}{C_0^{\theta}}\\
&-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Bigg)+\frac{rC'_0L_0^3}{C_0\Delta_0^2}\left(\frac{2a}{A_0}+\frac{c'}{C'_0}-\frac{c}{C_0}+\frac{3l}{L_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{L_0A_0^4B_0^2C'_0r^3}{C_0\Delta_0^2}\\
&\times\left(\frac{l}{L_0}+\frac{4a}{A_0}+\frac{2b}{B_0}+\frac{c'}{C'_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&-\frac{B'_0C'_0L_0^5}{B_0^3C_0\Delta_0^2}\Bigg(\frac{b'}{B'_0}+\frac{c'}{C'_0}+\frac{5l}{L_0}\\
&-\frac{3b}{B_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Bigg)+\frac{A_0^2L_0^3}{\Delta_0^2}\left(\frac{2a}{A_0}+\frac{3l}{L_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&-\frac{L_0^3L'_0}{B_0^2\Delta_0^2}\left(\frac{3l}{L_0}+\frac{2l'}{L'_0}-\frac{2\Delta_p}{\Delta_0}-\frac{2b}{B_0}\right)-\frac{br^2B_0A_0^2L_0^2C_0^{\theta}}{C_0\Delta_0^2}\\
&+\frac{r^4A_0^3B_0^2A'_0L'_0}{2L_0^2}\Bigg(\frac{2b}{B_0}+\frac{3a}{A_0}\\
&+\frac{a'}{A'_0}+\frac{l}{L'_0}-\frac{2\Delta_p}{\Delta_0}\Bigg)+\frac{4r^4B_0A_0^4B'_0L'_0}{\Delta_0^2}\\
&\times\left(\frac{3l}{L_0}+\frac{a}{A_0}+\frac{a^{\theta}}{A_0^{\theta}}+\frac{b^{\theta}}{B_0^{\theta}}-\frac{b}{B_0}\right)\\
&-\frac{3rA_0^2L_0^2L'_0}{2\Delta_0^2}\left(\frac{2a}{A_0}+\frac{l'}{L'_0}+\frac{2l}{L_0}-\frac{2\Delta_p}{\Delta_0}\right)\\
&+\frac{2r^3A_0^4B_0^2L'_0}{\Delta_0^2}\\
&\times\left(\frac{4a}{A_0}+\frac{2b}{B_0}+\frac{l'}{L'_0}-\frac{2\Delta_p}{\Delta_0}\right)+\frac{L_0^3A_0A_0^{\theta}C_0^{\theta}}{C_0\Delta_0^2}\\
&\times\Bigg(\frac{3l}{L_0}+\frac{a}{A_0}+\frac{a^{\theta}}{A_0^{\theta}}+\frac{c^{\theta}}{C_0^{\theta}}\\
&-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Bigg)-\frac{r^4A_0^4B_0^2L'_0C'_0T}{C_0\Delta_0^2}\Bigg(\frac{l'}{L'_0}+\frac{c'}{C'_0}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4a}{A_0}+\frac{2b}{B_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Big)\\
&-\frac{L_0'B_0'L_0^4}{2B_0^3\Delta_0^2}\Big(\frac{l'}{L_0'}+\frac{b'}{B_0'}+\frac{4l}{L_0}-\frac{3b}{B_0}-\frac{2\Delta_p}{\Delta_0}\Big)\\
&+\frac{2L_0'C_0'L_0^4}{B_0^2C_0\Delta_0^2}\Big(\frac{l'}{L_0'}+\frac{c'}{C_0'}\\
&+\frac{4l}{L_0}-\frac{2b}{B_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Big)-\frac{r^4A_0^4B_0^2L_0''}{2\Delta_0^2}\\
&\times\Big(\frac{c}{C_0}+\frac{l''}{L_0''}+\frac{2b}{B_0}+\frac{4a}{A_0}-\frac{2\Delta_p}{\Delta_0}\Big)\\
&+\frac{L_0''L_0^4}{2\Delta_0^2B_0^2}\Big(\frac{l''}{L_0''}+\frac{4l}{L_0}-\frac{2\Delta_p}{\Delta_0}-\frac{2b}{B_0}\Big)\\
&+\frac{C_0''L_0^5}{\Delta_0^2B_0^2C_0}\Big(\frac{c''}{C_0''}+\frac{5l}{L_0}-\frac{2b}{B_0}-\frac{c}{C_0}\\
&-\frac{2\Delta_p}{\Delta_0}\Big)+\frac{A_0^2L_0^3C_0^{\theta\theta}}{C_0\Delta_0^2}\Big(\frac{2a}{A_0}+\frac{c^{\theta\theta}}{C_0^{\theta\theta}}+\frac{3l}{L_0}-\frac{2\Delta_p}{\Delta_0}-\frac{c}{C_0}\Big)\\
&+\frac{r^2A_0A_0''L_0^3}{\Delta_0^2}\\
&\times\Big(\frac{a}{A_0}+\frac{a''}{A_0''}+\frac{3l}{L_0}-\frac{\Delta_p}{\Delta_0}\Big)+\frac{r^2A_0^2B_0''L_0^3}{B_0\Delta_0^2}\\
&\times\Big(\frac{2a}{A_0}+\frac{b''}{B_0''}+\frac{3l}{L_0}-\frac{b}{B_0}-\frac{\Delta_p}{\Delta_0}\Big)\\
&+\frac{r^4A_0^3B_0^2L_0A_0''}{\Delta_0^2}\Big(\frac{l}{L_0}+\frac{2b}{B_0}+\frac{3a}{A_0}+\frac{a''}{A_0''}-\frac{\Delta_p}{\Delta_0}\Big)\\
&+\frac{r^2A_0^4B_0B_0^\theta}{\Delta_0^2}\Big(\frac{l}{L_0}+\frac{4a}{A_0}+\frac{b}{B_0}+\frac{b^\theta}{B_0^\theta}-\frac{\Delta_p}{\Delta_0}\Big)\\
&+\frac{L_0A_0^4B_0r^4B_0''}{\Delta_0^2}\\
&\times\Big(\frac{l}{L_0}+\frac{4a}{A_0}-\frac{b}{B_0}+\frac{b''}{B_0''}-\frac{\Delta_p}{\Delta_0}\Big)\\
&+\frac{A_0^2L_0^3L_0^{\theta\theta}}{B_0\Delta_0^2}\Big(\frac{2a}{A_0}+\frac{b^{\theta\theta}}{L_0^{\theta\theta}}+\frac{3l}{L_0}-\frac{b}{B_0}-\frac{\Delta_p}{\Delta_0}\Big)\\
&+\frac{2C_0''A_0^2r^2L_0^3}{C_0\Delta_0^3}\Big(\frac{c''}{C_0''}-\frac{\Delta_p}{\Delta_0}-\frac{c}{C_0}+\frac{2a}{A_0}+\frac{3l}{L_0}\Big)\\
&+\frac{L_0C_0''B_0^2A_0^4r^4}{C_0\Delta_0^2}\Big(\frac{l}{L_0}+\frac{c''}{C_0''}+\frac{2b}{B_0}+\frac{4a}{A_0}-\frac{c}{C_0}-\frac{2\Delta_p}{\Delta_0}\Big)\\
&+\frac{4L_0A_0^4B_0^2r^2C_0^{\theta\theta}}{C_0\Delta_0^2}\Big(\frac{l}{L_0}+\frac{4a}{A_0}+\frac{2b}{B_0}+\frac{c^{\theta\theta}}{C_0^{\theta\theta}}-\frac{c}{C_0}-\frac{\Delta_p}{\Delta_0}\Big)\\
=&-\Big[\rho_{0(m\phi)}\\
&\times\Big\{\frac{b}{B_0}+\frac{c}{C_0}+\frac{1}{\Delta_0}\Big(r^2aA_0^2B_0^2+lL_0+r^2bB_0L_0\Big)\Big\}\\
&+(\rho_{0(m\phi)}+p_{0(m\phi)})\\
&\times\frac{A_0^2B_0^2}{\Delta_0^2}\Big\{r^2\Big(\frac{2b}{B_0}+\frac{2c}{C_0}\Big)+\frac{A_0^2}{A_0^2B_0^2}\Big(\frac{b}{B_0}+\frac{l}{L_0}-\frac{a}{A_0}+\frac{c}{C_0}\Big)\Big\}\\
&+\frac{\Pi_{I0(m\phi)}}{3}\Big(\frac{b}{B_0}-\frac{c}{C_0}\Big)+\frac{\Pi_{II0(m\phi)}}{3\Delta_0}\\
&\times\Big\{r^2A_0^2B_0^2\Big(\frac{b}{B_0}-\frac{c}{C_0}\Big)+L_0^2\Big(\frac{l}{L_0}-\frac{a}{A_0}-\frac{c}{C_0}\Big)\Big\}\Big]\\
&+(x_{3(p3)}+y_{3(p3)}).
\end{aligned}\tag{A6}
$$

Here the scalar field stress with subscript $p1$, $p2$ indicates perturbed values coupled to w^2, w, respectively, and $p3$ shows a situation otherwise determined. The values of $F_{(m\phi)}$, $\bar{E}_{0(a)}$, and $\bar{E}_{0(b)}$ given in Eq. (25) are as follows:

$$
\begin{aligned}
F_{(m\phi)}=&-\Big[\rho_{0(m\phi)}\Big\{\frac{b}{B_0}+\frac{c}{C_0}+\frac{1}{\Delta_0}\\
&\times\Big(r^2aA_0B_0^2+lL_0+r^2bB_0A_0^2\Big)\Big\}\\
&+(\rho_{0(m\phi)}+p_{0(m\phi)})\frac{A_0^2B_0^2}{L_0}\Big\{r^2\Big(\frac{2b}{B_0}+\frac{2c}{C_0}\Big)\\
&+\frac{L_0^2}{A_0^2B_0^2}\Big(\frac{b}{B_0}+\frac{l}{L_0}-\frac{a}{A_0}\\
&+\frac{c}{C_0}\Big)\Big\}+\frac{\Pi_{I0(m\phi)}}{3}\Big(\frac{b}{B_0}-\frac{c}{C_0}\Big)\\
&+\frac{\Pi_{II0(m\phi)}}{3L_0}\Big\{r^2A_0^2B_0^2\Big(\frac{b}{B_0}-\frac{c}{C_0}\Big)\\
&+L_0^2\Big(\frac{l}{L_0}-\frac{a}{A_0}-\frac{c}{C_0}\Big)\Big\}\Big],
\end{aligned}\tag{A7}
$$

$$
\begin{aligned}
\bar{E}_{0(a)}=&(\dot{v}_{1(ap)}+\dot{w}_{1(ap)})+(x'_{1(ap)}+y'_{1(ap)})\\
&+(x^\theta_{3(ap)}+y^\theta_{3(ap)})+\Big(\frac{c}{C_0}\\
&+\frac{b}{B_0}+\frac{2A_0B_0ar^2}{\Delta_0}+\frac{B_0A_0^2b}{\Delta_0}+\frac{2lL_0}{\Delta_0}\Big)(v_{1(a0)}+w_{1(a0)})\\
&+\Big(\frac{C_0'}{C_0}+\frac{B_0'}{B_0}+\frac{2B_0^3r^2A_0A_0'}{\Delta_0}+\frac{2L_0L_0'}{\Delta_0}\Big)(x_{1(a0)}+y_{1(a0)})\\
&+\frac{bB_0^2L_0r^2}{\Delta_0}+\Big(\frac{C_0^\theta}{C_0}+\frac{B_0^\theta}{B_0}\\
&+\frac{\Delta_0^\theta}{\Delta_0}\Big)(v_{(20)}+w_{(20)})\Big(\frac{B_0^2r^2}{\Delta_0}\Big)\\
&+(x_{3(ap)}+y_{3(ap)})+(v_{2(ap)}+w_{2(ap)})\\
&\times\Big(\frac{-2L_0B_0}{\Delta_0}\Big)+\Big[\frac{-2B_0^2r^2L_0^\theta}{\Delta_0}+\frac{L_0^2B_0r^2B_0^\theta}{\Delta_0}\Big]\\
&\times(x_{3(ap)}+y_{3(ap)})+(x_{3(a0)}\\
&+y_{3(a0)})\Big(\frac{B_0^2r^4b}{\Delta_0}\Big)+(x_{4(a0)}+y_{4(a0)})\Big(\frac{C_0B_0^2r^2c}{\Delta_0}\Big)\\
&-\frac{2L_0C_0^\theta}{\Delta_0}(x_{4(ap)}\\
&+y_{4(ap)})+(x_{(2ap)}+y_{(2ap)})\Big[\frac{-B_0^2r^2L_0'}{\Delta_0}+\frac{L_0r^2B_0B_0'}{\Delta_0}\\
&+\frac{L_0B_0^2r^2}{\Delta_0}\Big],
\end{aligned}\tag{A8}
$$

$$
\begin{aligned}
\bar{E}_{0(b)}=&\Big[\frac{C_0^\theta}{C_0}+\frac{B_0^\theta}{B_0}+\frac{\Delta_0^\theta}{\Delta_0}+\frac{2B_0^2r^2A_0A_0'}{\Delta_0}\Big](x_{3(bp)}+y_{3(bp)})\\
&+(v_{2(bp)}+w_{2(bp)})\Big[\frac{-2L_0B_0^\theta}{\Delta_0}\Big]+\Big[\frac{-2B_0^2r^2L_0^\theta}{\Delta_0}\\
&+\frac{L_0B_0r^2L_0B_0^\phi}{\Delta_0}\Big](x_{3(bp)}
\end{aligned}
$$

$$+y_{3(bp)})+(v_{2(b0)}+w_{2(b0)})\left[\frac{l}{L_0}+\frac{b}{B_0}+\frac{\Delta_0'}{\Delta_0}\right]$$
$$-\frac{2L_0C_0^\theta}{\Delta_0}(v_{(4bp)}+w_{(4bp)})$$
$$+(x_{2(bp)}+y_{2(bp)})\left(-B_0^2r'L_0'+\frac{r^2B_0^2B_0'}{\Delta_0}+\frac{L_0B_0^2r^2}{\Delta_0}\right)$$
$$+(x_{2(0)}+y_{2(0)})\Big(-B_0^2$$
$$\times r^2L_0'\left(\frac{b^\theta}{B^\theta}+\frac{l'}{L_0'}\right)+L_0r^3B_0B_0'\left(\frac{b'}{B_0'}+\frac{l}{L_0}\right)$$
$$+L_0B_0^2r^2\left(\frac{l}{L_0}+\frac{2b}{B_0}\right)\Big)$$
$$+(x^\theta_{3(bp)}+y^\theta_{3(bp)}). \quad \text{(A9)}$$

The subscripts $a0$ and ap denote the scalar field with unperturbed as well as perturbed terms coupled to the frequency, while $b0$ and bp show scalar field static and perturbed configurations otherwise determined. The value of the scalar field term $\bar{E}_1$ is given by

$$\bar{E}_1=(\dot{x}_{(1p)}+\dot{y}_{(1p)})+(v'_{1(p)}+w'_{1(p)})+(x_{2(p)}+y_{2(p)})$$
$$+(v_{1(0)}+w_{1(0)})^\theta\frac{A_0A_0'}{B_0^2}$$
$$\times e^{iwt}\left(2\frac{b}{B_0}+\frac{a}{A_0}+\frac{a'}{A_0'}\right)+(v_{1(p)}+w_{1(p)})\frac{A_0A_0'}{B_0^2}$$
$$+(v_{2(0)}+w_{2(0)})e^{iwt}$$
$$\times\left[\frac{2b'}{B_0B_0'}-\frac{B_0'b}{B_0^2}\right]+(v_{2(0)}+w_{2(0)})e^{iwt}$$
$$\times\Bigg[\frac{B_0^2r^2A_0A_0'}{\Delta_0}\left(\frac{2b}{B_0^2}+\frac{a}{A_0}+\frac{a'}{A_0'}-\frac{\Delta_p}{\Delta_0}\right)$$
$$+\frac{2L_0L_0'}{\Delta_0}\left(\frac{l}{L_0}+\frac{l'}{L_0'}+\frac{\Delta_p}{\Delta_0}\right)+\frac{A_0^2B_0B_0'r^2}{\Delta_0}$$
$$\times\left(2\frac{a}{A_0}+\frac{b}{B_0}+\frac{b'}{B_0'}-\frac{\Delta_p}{\Delta_0}\right)+\frac{c^\theta}{C_0}$$
$$+\frac{A_0^2B_0^2r}{\Delta_0}\left[\frac{2a}{A_0}+2\frac{b}{B_0}-\frac{\Delta_p}{\Delta_0}\right]\Bigg]$$
$$+(v_{2(p)}+w_{2(p)})\Bigg[2\frac{B_0'}{B_0}+\frac{B_0^2r^2A_0A_0'}{\Delta_0}+\frac{2L_0L_0'}{\Delta_0}$$
$$+\frac{A_0^2B_0r^2B_0'}{\Delta_0}\Bigg]+(v_{3(p)}+w_{3(p)})\left(-r^2\frac{B_0'}{B_0}-r\right)$$
$$-(v_{4(p)}+w_{2(p)})\frac{C_0C_0'}{B_0^2}$$
$$+(v_{3(0)}+w_{3(0)})e^{iwt}\left(\frac{rb'}{B_0}-\frac{b}{B\,0}\right)$$
$$-(v_{4(p)}+w_{4(p)})\left[\frac{C_0'c}{C_0}+\frac{c'}{C_0}-\frac{2b}{B_0}\right]$$
$$+\left(\frac{-L_0A_0A_0^\theta}{\Delta_0}+\frac{A_0B_0r^2B_0^\theta}{\Delta_0}\right)(x_{1(p)}+y_{1(p)})$$
$$+e^{iwt}\Bigg[3\frac{L_0'}{B_0^2}\left(\frac{l'}{L_0'}-\frac{2b}{B_0}\right)$$
$$+\frac{B_0'}{B_0}\left(\frac{b'}{B_0'}-\frac{b}{B_0}\right)+\frac{L_0L_0'}{\Delta_0}\left(\frac{l}{L_0}+\frac{l'}{L_0'}-\frac{\Delta_p}{\Delta_0}\right)$$
$$+\frac{A_0^2B_0r^2B_0'}{\Delta_0}\Big(\frac{2a}{A_0}$$
$$+\frac{b}{B_0}+\frac{b'}{B_0'}-\frac{\Delta_p}{\Delta_0}\Big)+\frac{A_0^2B_0^2r}{\Delta_0}\left[\frac{2a}{A_0}+\frac{2b}{B_0}\right]\Bigg]$$
$$\times(x_{2(0)}+y_{2(0)})+(x_{2(p)}$$
$$+y_{2(p)})\left[3\frac{L_0'}{B_0^2}+\frac{B_0'}{B_0}+\frac{L_0L_0'}{\Delta_0}+\frac{A_0^2B_0r^2B_0'}{\Delta_0}+\frac{A_0^2B_0^2r}{\Delta_0}\right] \quad \text{(A10)}$$

The values of $\delta_{(m(\mathrm{BD}))}$ and $\lambda_{(m(\mathrm{BD}))}$ are

$$\delta_{(m(\mathrm{BD}))}=\frac{1}{B_0^2}\Bigg(\frac{p_{0(m\phi)}\beta}{\rho_{0(m\phi)}+p_{0(m\phi)}}+\frac{4\Pi_{I0(m\phi)}\beta}{9(\rho_{0(m\phi)}+\Pi_{I0(m\phi)})}$$
$$+\frac{2\Pi_{II0(m\phi)}\beta}{9(\rho_{0(m\phi)}+\Pi_{II0(m\phi)})}\Bigg)'-\Bigg[\frac{p_0}{\rho_{0(m\phi)}+p_{0(m\phi)}}$$
$$+\frac{4\Pi_{I0(m\phi)}}{9(\rho_{0(m\phi)}+\Pi_{I0(m\phi)})}$$
$$+\frac{2\Pi_{II0(m\phi)}}{9(\rho_{0(m\phi)}+\Pi_{II0(m\phi)})}\Bigg]\frac{\beta}{B_0^2}\Bigg\{\frac{C_0'}{C_0}+\frac{3L_0L_0'}{2\Delta_0}$$
$$+\frac{r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\Bigg\}$$
$$+\frac{r^2A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\left(\frac{\Pi_{k\chi0(m\phi)}\beta}{\rho_{0(m\phi)}+\Pi_{k\chi0(m\phi)}}\right)^\theta+\frac{\Pi_{II0(m\phi)}\beta}{\rho_{0(m\phi)}+\Pi_{II0(m\phi)}}$$
$$\times\frac{r^2A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\times\Bigg\{\frac{A_0^\theta}{A_0}+\frac{6B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}+\frac{4L_0L_0^\theta}{\Delta_0}$$
$$+\frac{4r^2A_0B_0^2}{\Delta_0}\left(\frac{A_0^\theta}{A_0}+\frac{B_0^\theta}{B_0}\right)\Bigg\}, \quad \text{(A11)}$$

$$\lambda_{\mathrm{mBD}}=\Bigg[\frac{\beta r^4A_0^4}{\Delta_0^2}\left(\frac{A_0'}{A_0}-\frac{L_0A_0^\theta}{r^2A_0B_0^2}\right)$$
$$\times\frac{2b}{B_0^3}\left\{p_0'+\frac{2}{9}(2\Pi'_{I0(m\phi)}+\Pi'_{II0(m\phi)})\right\}$$
$$-\Bigg[\left\{\left(\frac{c}{C_0}\right)'+\frac{3L_0L_0'}{2\Delta_0}\left(\frac{l}{L_0}+\frac{l'}{L_0'}-\frac{\Delta_p}{\Delta_0}\right)\right\}$$
$$+\frac{r^2A_0^2B_0^2}{\Delta_0^2}\left(\frac{2a}{A_0}+\frac{2b}{B_0}-\frac{\Delta_p}{\Delta_0}\right)\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)$$
$$+\frac{r^2A_0^2B_0^2}{\Delta_0}\left(\frac{a}{A_0}+\frac{b}{B_0}\right)'\Bigg]$$
$$\times\frac{1}{B_0^2}\Big\{p_{0(m\phi)}+\frac{2}{9}(2\Pi_{I0(m\phi)}$$
$$+\Pi_{II0(m\phi)})\Big\}-\frac{2b}{B_0^2}\left\{p_{0(m\phi)}+\frac{2}{9}(2\Pi_{I0(m\phi)}+\Pi_{II0(m\phi)})\right\}$$
$$\times\Bigg\{\frac{C_0'}{C_0}+\frac{3L_0L_0'}{2\Delta_0}$$
$$+\frac{r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\Bigg\}+\frac{r^2A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\Pi^\theta_{k\chi0(m\phi)}$$
$$\times\left(\frac{a}{A_0}+\frac{3b}{B_0}-\frac{3\Delta_p}{\Delta_0}\right)$$

$$+\frac{r^2A_0B_0^3\Pi_{k\chi0(m\phi)}}{\Delta_0^{\frac{3}{2}}}\left(\frac{a}{A_0}+\frac{3b}{B_0}-\frac{3\Delta_p}{\Delta_0}\right)$$
$$\times\left\{\frac{A_0^\theta}{A_0}+\frac{6B_0^\theta}{B_0}+\frac{C_0^\theta}{C_0}+\frac{4L_0L_0^\theta}{\Delta_0}\right.$$
$$\left.+\frac{4r^2A_0^2B_0^2}{\Delta_0}\left(\frac{A_0^\theta}{A_0}+\frac{B_0^\theta}{B_0}\right)\right\}+\frac{r^2A_0B_0^3}{\Delta_0^{\frac{3}{2}}}\Pi_{k\chi0(m\phi)}$$
$$\times\left[\frac{6B_{0\theta}}{B_0}\left(\frac{b_\theta}{B_{0\theta}}+\frac{b}{B_0}\right)\right.$$
$$\times\left(\frac{a}{A_0}+\frac{c}{C_0}\right)^\theta+\frac{4L_0L_{0\theta}}{\Delta_0}\left(\frac{l}{L_0}+\frac{l^\theta}{L_0^\theta}-\frac{\Delta_p}{\Delta_0}\right)$$
$$+\frac{4r^2A_0^2B_0^2}{\Delta_0}\left(\frac{2a}{A_0}+\frac{2b}{B_0}\right.$$
$$\left.\left.-\frac{\Delta_p}{\Delta_0}\right)\left(\frac{a}{A_0}+\frac{b}{B_0}\right)^\theta\right]-\frac{\rho_{0(m\phi)}r^4A_0^4}{\Delta_0^2}\left\{\left(\frac{a}{A_0}\right)'\right.$$
$$-\frac{L_0}{r^2}\frac{A_0^\theta}{A_0B_0^2}\left(\frac{l}{L_0}+\frac{a^\theta}{A_0^\theta}\right.$$
$$\left.\left.-\frac{a}{A_0}-\frac{2b}{B_0}\right)\right\}+\frac{\rho_{0(m\phi)}L_0^2A_0^2r^2}{\Delta_0^2}\left(\frac{2l}{L_0}+\frac{2a}{A_0}-\frac{2\Delta_p}{\Delta_0}\right)$$
$$+\frac{\rho_{0(m\phi)}L_0^2A_0^2r^2}{\Delta_0^2}$$
$$\left.\times\left(\frac{l}{L_0}+\frac{b}{B_0}\right)'-L_0^2r^2\frac{A_0^2}{\Delta_0^2}\beta\left\{\frac{L_0'}{2L_0}+\frac{1}{r}+\frac{B_0'}{B_0}\right\}\right]. \tag{A12}$$

Here $\beta=\left[(F_{(m\phi)}+\bar{E}_{0(a)})iw+\bar{E}_{0(b)}\right]$.

The values of $\delta_{(m(\mathrm{BD}))N}$, $\lambda_{(m(\mathrm{BD}))N}$, and $\bar{E}_{1(N)}$ are

$$\delta_{(m(\mathrm{BD}))N}=\left[\beta_{(N)}\left\{p_{0(m\phi)}+\frac{2}{9}(2\Pi_{I0(m\phi)}+\Pi_{II0(m\phi)})\right\}\right]'$$
$$+\frac{11\beta_{(N)}}{4r}\left\{p_{0(m\phi)}+\frac{2}{9}(2\Pi_{I0(m\phi)}\right.$$
$$\left.+\Pi_{II0(m\phi)})\right\}-\left[\frac{\Pi_{k\chi0(m\phi)}}{2\sqrt{2}r}\beta_{(N)}\right]^\theta, \tag{A13}$$
$$\lambda_{(m(\mathrm{BD}))N}=\left[\left\{p_{0(m\phi)}+\frac{2}{9}(2\Pi_{I0(m\phi)}+\Pi_{II0(m\phi)})\right\}\right.$$
$$\times\left\{\frac{1}{2}(a+b)'+\left(\frac{c}{r}\right)'+\frac{1}{2r}\left(2a+11b-\frac{\Delta_p}{2r^2}\right)\right\}$$
$$+\frac{\Pi_{k\chi0(m\phi)}}{2\sqrt{2}}\left[2(a+b)^\theta\left(2a+2b\right.\right.$$
$$\left.\left.-\frac{\Delta_p}{2r^2}\right)\right]\frac{3}{4r}\left(l'+\frac{l}{r}-\frac{\Delta_p}{2r^2}\right)\left\{p_{0(m\phi)}\right.$$
$$\left.\left.+\frac{2}{9}(2\Pi_{I0(m\phi)}+\Pi_{II0(m\phi)})\right\}\right]$$
$$+\frac{\rho_{0(m\phi)}}{4}\left(2b'-a'+\frac{7}{2r}+\frac{6c}{r}-\frac{\Delta_p}{r^2}\right), \tag{A14}$$
$$\bar{E}_{1(N)}=(v_{2(0)N}+w_{2(0)N})\left[\frac{2}{r}\left[\frac{l}{r}+l'+\frac{\Delta_p}{r^2}\right]\right]$$
$$-(v_{3(0)N}+w_{3(0)N})\frac{r^2b}{B_0}$$
$$-(v_{4(0)N}+w_{4(0)N})\left[c'-\frac{2b}{B_0}\right]$$
$$+(x_{2(0)N}+y_{2(0)N})\left[\frac{l}{r^2}+\frac{l'}{r}-\frac{\Delta_p}{r^3}\right]$$
$$+\frac{1}{r}[a+2b], \tag{A15}$$

where

$$\beta_{(N)}=\left((3b+\frac{2c}{r}+\frac{l}{r})+(v_{1(0)N}+w_{1(0)N})\right.$$
$$\left.+r^2b(x_{3(0)N}+y_{3(0)N})\right)iw$$
$$+(v_{2(0)N}+w_{2(0)N})\left[\frac{l}{r}+b+\frac{2}{r}\right]+(x_{2(0)N}$$
$$+y_{2(0)N})\left[\frac{l}{r}+\frac{2b}{B_0}\right].$$

In the pN approximations, the values of $\delta_{(m(\mathrm{BD}))\mathrm{pN}}$, $\lambda_{(m(\mathrm{BD}))\mathrm{pN}}$, and $\bar{E}_{1(\mathrm{pN})}$ are

$$\delta_{(m(\mathrm{BD}))\mathrm{pN}}=-\left(1-\frac{2m_0}{r_1}\right)\left(\frac{p_{0(m\phi)}\beta_{(\mathrm{pN})}}{\rho_{0(m\phi)}+p_{0(m\phi)}}\right.$$
$$+\frac{4\Pi_{I0(m\phi)}\beta_{(\mathrm{pN})}}{9(\rho_{0(m\phi)}+\Pi_{I0(m\phi)})}$$
$$\left.+\frac{2\Pi_{II0(m\phi)}\beta_{(\mathrm{pN})}}{9(\rho_{0(m\phi)}+\Pi_{II0(m\phi)})}\right)'-\left\{\frac{p_{0(m\phi)}}{\rho_{0(m\phi)}+p_{0(m\phi)}}\right.$$
$$+\frac{4\Pi_{I0(m\phi)}}{9(\rho_{0(m\phi)}+\Pi_{I0(m\phi)})}$$
$$\left.+\frac{2\Pi_{II0(m\phi)}}{9(\rho_{0(m\phi)}+\Pi_{II0(m\phi)})}\right\}\left(1-\frac{2m_0}{r_1}\right)\beta_{(\mathrm{pN})}$$
$$\times\left\{\frac{7}{4r}+\frac{1}{2}\left(1-\frac{4m_0^2}{r_1^2}\right)\right.$$
$$\left.\times\left(1-\frac{m_0^2}{r_1^2}+\frac{1}{r}\right)\right\}+\frac{1}{2\sqrt{2}r}\left(1-\frac{m_0}{r_1}\right)\left(1+\frac{3m_0}{r_1}\right)$$
$$\times\left(\frac{\Pi_{k\chi0(m\phi)}\beta_{(\mathrm{pN})}}{\rho_{0(m\phi)}+\Pi_{k\chi0(m\phi)}}\right)^\theta, \tag{A16}$$
$$\lambda_{(m(\mathrm{BD}))\mathrm{pN}}=\left[\frac{b\beta_{(\mathrm{pN})}}{2}\left(1-\frac{4m_0}{r_1}\right)\left(1+\frac{m_0}{r_1}\right)\right.$$
$$\times\left(1-\frac{3m_0}{r_1}\right)\left(1-\frac{m_0}{r_1}\right)'$$
$$\times\left\{p_{0(m\phi)}'+\frac{2}{9}(2\Pi'_{I0(m\phi)}+\Pi'_{II0(m\phi)})\right\}-\left[\left\{\left(\frac{c}{r}\right)'\right.\right.$$
$$\left.+\frac{3}{4r}\left(\frac{l}{r}+l'-\frac{\Delta_p}{2r^2}\right)\right\}$$
$$+\frac{1}{2r^2}\left(1-\frac{4m_0^2}{r_1^2}\right)\left(2a\left(1+\frac{m_0}{r_1}\right)\right.$$
$$+2b\left(1-\frac{m_0}{r_1}\right)-\frac{\Delta_p}{2r^2}\right)\left(\left(1-\frac{m_0}{r_1}\right)'\right.$$
$$\left.\times\left(1+\frac{m_0}{r_1}\right)+\frac{1}{r}\right)+\frac{1}{2}\left(a\left(1+\frac{m_0}{r_1}\right)\right.$$
$$\left.\left.+b\left(1-\frac{m_0}{r_1}\right)\right)'\right]\left\{p_{0(m\phi)}\right.$$
$$\left.+\frac{2}{9}(2\Pi_{I0(m\phi)}+\Pi_{II0(m\phi)})\right\}\left(1-\frac{2m_0}{r_1}\right)$$
$$-2b\left(1-\frac{2m_0}{r_1}\right)\left\{p_{0(m\phi)}\right.$$

$$+\frac{2}{9}(2\Pi_{I0(m\phi)}+\Pi_{II0(m\phi)})\Big\}\Big\{\frac{7}{4r}$$
$$+\frac{1}{2}\left(1-\frac{4m_0^2}{r_1^2}\right)\left(\left(1-\frac{m_0}{r_1}\right)'\right.$$
$$\times\left(1+\frac{m_0}{r_1}\right)\left.+\frac{1}{r}\right)\Big\}+\frac{1}{2\sqrt{2}r}\Pi^{\theta}_{k\chi0(m\phi)}$$
$$\times\left(1-\frac{m_0}{r_1}\right)\left(1+\frac{3m_0}{r_1}\right)$$
$$\times\left(a\left(1+\frac{m_0}{r_1}\right)+3b\left(1-\frac{m_0}{r_1}\right)-\frac{3\delta_p}{2r^2}\right)$$
$$+\frac{\Pi_{k\chi0(m\phi)}}{2\sqrt{2}r}\Big[6\left(1-\frac{m_0}{r_1}\right)$$
$$\times\left(1+\frac{m_0}{r_1}\right)^{\theta}\left(b^{\theta}\left(1-\frac{m_0}{r_1}\right)^{\theta}\right.$$
$$\left.+b\left(1-\frac{m_0}{r_1}\right)\right)\left(a\left(1+\frac{m_0}{r_1}\right)+\frac{c}{r}\right)^{\theta}$$
$$+2\left(1-\frac{4m_0^2}{r_1^2}\right)+\left(2a\left(1+\frac{m_0}{r_1}\right)\right.$$
$$\left.+2b\left(1-\frac{m_0}{r_1}\right)-\frac{\delta_p}{r^2}\right)$$
$$\times\left(a\left(1+\frac{m_0}{r_1}\right)+b\left(1-\frac{m_0}{r_1}\right)\right)^{\theta}\Big]$$
$$-\frac{\rho_{0(m\phi)}}{4}\left(1-\frac{4m_0}{r_1}\right)$$
$$\times\Big\{\left(a\left(1+\frac{m_0}{r_1}\right)\right)'-\frac{1}{r}\left(1-\frac{m_0}{r_1}\right)^{\theta}$$
$$\times\left(1+\frac{m_0}{r_1}\right)\left(1-\frac{2m_0}{r_1}\right)\left(\frac{l}{r}+a^{\theta}\left(1+\frac{m_0}{r_1}\right)^{\theta}\right.$$
$$\left.\times-a\left(1+\frac{m_0}{r_1}\right)-2b\left(1-\frac{m_0}{r_1}\right)\right)\Big\}$$
$$+\frac{\rho_{0(m\phi)}}{2\sqrt{2}r}$$
$$\times\left(1-\frac{2m_0}{r_1}\right)\left(\frac{2l}{r}+2a\left(1+\frac{m_0}{r_1}\right)-\frac{\Delta_p}{r^2}\right)$$
$$+\frac{\mu_0}{2\sqrt{2}}\left(1-\frac{2m_0}{r_1}\right)\left(\frac{l}{r}+b\right.$$
$$\left.\times\left(1-\frac{m_0}{r_1}\right)\right)'-\frac{\beta_p}{2\sqrt{2}}\left(1-\frac{2m_0}{r_1}\right)\Big\{\frac{3}{2r}$$
$$+\left(1+\frac{m_0}{r_1}\right)'\left(1-\frac{m_0}{r_1}\right)\Big\}\Big],\tag{A17}$$

$$\bar{E}_{1(\mathrm{pN})}=\Big[(\dot{x}^{(\mathrm{pN})}_{1(ap)}+\dot{y}^{(\mathrm{pN})}_{1(ap)})+(v'^{(\mathrm{pN})}_{2(ap)}+w'^{(\mathrm{pN})}_{2(ap)})$$
$$\times+(x^{\theta(\mathrm{pN})}_{2(ap)}+y^{\theta(\mathrm{pN})}_{2(ap)})$$
$$\times+(v^{(\mathrm{pN})}_{1(ap)}+w^{(\mathrm{pN})}_{1(ap)})(\frac{m_0}{r})'+(v^{(\mathrm{pN})}_{2(ap)}$$
$$\times+w^{(\mathrm{pN})}_{2(ap)})\left[4(\frac{m_0}{r})'+\frac{l}{r}\right]\Big]iw+(v^{(\mathrm{pN})}_{2(bp)}$$
$$\times+w^{(\mathrm{pN})}_{2(bp)})'+(x^{(\mathrm{pN})}_{2(ap)}+y^{(\mathrm{pN})}_{2(ap)})^{\theta}+(v^{(\mathrm{pN})}_{1(0p)}$$
$$\times+w^{(\mathrm{pN})}_{2(0p)})\left(\frac{m_0}{r}\right)',\tag{A18}$$

where
$$\beta_{(\mathrm{pN})}=((F_{(m\phi)\mathrm{pN}}+\bar{E}_{0(a)\mathrm{pN}})iw+\bar{E}_{0(b)\mathrm{pN}}).$$

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