

Isothermal plasma wave properties of the Schwarzschild de-Sitter black hole in a Veselago medium

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Abstract In this paper, we study wave properties of isothermal plasma for the Schwarzschild de-Sitter black hole in a Veselago medium. We use ADM 3 + 1 formalism to formulate general relativistic magnetohydrodynamical (GRMHD) equations for the Schwarzschild de-Sitter spacetime in Rindler coordinates. Further, Fourier analysis of the linearly perturbed GRMHD equations for the rotating (non-magnetized and magnetized) background is taken whose determinant leads to a dispersion relation. We investigate wave properties by using graphical representation of the wave vector, the refractive index, change in refractive index, phase and group velocities. Also, the modes of wave dispersion are explored. The results indicate the existence of the Veselago medium.

Keywords 3 + 1 formalism · SdS black hole · Veselago medium · GRMHD equations · Isothermal plasma · Dispersion relations

1 Introduction

Our solar system is filled with a wide range of celestial objects. Black hole is one of such objects, having so strong gravitational pull that no nearby matter or radiation, not even light can escape from its gravitational field. Astronomers are curious to extract real life examples of black hole. The presence of matter in the form of white dwarfs and neutron stars

suggests the existence of stellar mass black holes. The accumulated evidence for the black hole existence is now very captivating. It is believed that collapse of a massive star under its own gravity leads to the formation of black hole (Das 2004). Plasmas are abundant in nature, almost found everywhere in an interstellar medium. It is a distinct state of matter with free electric charge carriers which behave collectively and respond strongly to electromagnetic fields (Raine and Thomas 2005). Black hole (in its surroundings) attracts plasma towards the event horizon due to its strong gravitational pull. The plasma forms an accretion disk due to interaction of plasma field with black hole gravity.

The theory of general relativistic magnetohydrodynamics is the most reliable discipline to examine the dynamics of magnetized plasma and effects of black hole gravity. The de-Sitter spacetime is a vacuum solution of the Einstein field equations including a positive cosmological constant (Rindler 2001). The Schwarzschild de-Sitter (SdS) metric describes a black hole expressing a patch of the de-Sitter spacetime. Since the SdS black hole is non-rotating, so plasma in magnetosphere moves only along the radial direction. According to the recent cosmological and astrophysical observations, our universe is accelerating rather than decelerating and inclusion of positive cosmological constant reveals the expanding universe (Reiss et al. 1998; Bahcall et al. 1999; Perlmutter et al. 1997). That is why our universe approaches to de-Sitter universe in future. This motivates the study of plasma waves in de-Sitter spacetime.

Peterson (1974) investigated the strong gravitational field close to the surface of compact objects for the Schwarzschild black hole. Narayan (2005) suggested that compact objects having mass three times the solar mass can be identified as black hole candidates. Plasma present in magnetosphere is perturbed by gravity of black hole. Zerilli (1970a, 1970b, 1970c) used linear perturbation to explore

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gravitational field of a particle falling in the Schwarzschild black hole. Price (1972a, 1972b) discussed dynamics of approximately spherical star by using non-spherical perturbations. Regge and Wheeler (1957) also used non-spherical perturbation to investigate the stability of Schwarzschild singularity. Gleiser et al. (1972) explored the stability of black holes by considering second order perturbations.

Arnowitt et al. (1962) (ADM) proposed 3 + 1 formalism for an easy approach to General Relativity (GR) by separating metric field into two parts (space and time) to characterize the coordinate system. Smarr and York (1978) used this formulation to explore spacetime kinematics numerically. Israel (1967, 1968) discussed event horizons in static vacuum and static electro-vacuum spacetimes. Thorne and Macdonald (1982a, 1982b) explained how 3 + 1 split is appropriate approach for black hole theory. Macdonald and Suen (1985) developed a self-consistent formalism to treat electromagnetic and gravitational fields near black hole horizon. Sakai and Kawata (1980) analyzed wave propagation in ultra-relativistic plasma, parallel to a constant magnetic field in a frame of two fluid model. Holcomb (1990) and Dettmann et al. (1993) constructed electrodynamical equations for the universe models. Holcomb and Tajima (1989) formulated linearized theory for relativistic plasma and found results for matter fluctuations in the early universe.

Rezolla et al. (2003) explored dynamics of thick disks around SdS black hole by considering the effects of cosmological constant. Font and Daigne (2002) studied stability of thick accretion disks around black holes. Myung (2001) developed entropy bounds for SdS black hole. Suneeta (2003) considered quasinormal modes for scalar field perturbations of SdS black hole. Setare (2005) obtained area and entropy spectrum near extremal SdS black hole horizon. Zhang (1989a) modified the stationary symmetric GRMHD black hole configuration theory. The same author (Zhang 1989b) explored the modes of perturbation in rotating black hole. Buzzi et al. (1995a, 1995b) determined the properties of waves propagating in two fluid plasma for the Schwarzschild black hole. Ali and Rahman (2009) explained transverse wave propagation in two fluid plasma around SdS black hole. Sharif and his collaborators (Sharif and Sheikh 2007a, 2007b, 2007c, 2008a, 2008b, 2008c, 2009a, 2009b; Sharif and Mustafa 2008; Sharif and Rafique 2010) have explored wave properties of cold, isothermal and hot plasmas with Schwarzschild as well as Kerr spacetimes in the usual medium.

The medium with both negative permeability and permittivity has the unusual electromagnetic properties named as Veselago medium or negative index medium (NIM), after a Russian physicist Veselago (1968). It is also called as double negative medium (DNM) or negative phase velocity medium (NPV). Valanju et al. (2002) presented treatment for refraction of electromagnetic waves in a NIM.

Ross et al. (2006) concluded that propensity of a rotating black hole is enhanced in the presence of charge to support wave propagation with negative phase velocity in its ergosphere. Ziolkowski and Heyman (2001) studied wave propagation analytically and numerically in NIM. Nagar et al. (2004) reported results from numerical simulations of gravitational radiations emitted due to matter accretion from non-rotating compact objects. In recent papers, Sharif and Mukhtar (2011a, 2011b) have discussed wave properties with non-rotating as well as rotating background plasmas (isothermal and hot) in this unusual medium.

This paper deals with wave properties of isothermal plasma around SdS black hole in a Veselago medium. We consider 3 + 1 GRMHD equations and determine a dispersion relation by Fourier analysis for both magnetized and non-magnetized backgrounds. The results are discussed by three dimensional plot of wave vector, refractive index and change in refractive index. The paper is organized as follows: In Sect. 2, linearly perturbed 3 + 1 GRMHD equations for isothermal plasma and their Fourier analysis is developed. Sections 3 and 4 provide reduced form of the GRMHD equations for rotating (non-magnetized and magnetized respectively) plasmas. We summarize our results in the last section.

2 GRMHD equations in a Veselago medium with isothermal plasma assumption

The general line element in ADM 3 + 1 formalism is given as follows (Zhang 1989b)

$$ds^2 = -\alpha^2 dt^2 + \eta_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt). \quad (2.1)$$

A natural observer associated with this spacetime is known as fiducial observer (FIDO), α denotes lapse function (ratio of FIDO proper time to universal time i.e., $\frac{dt}{dt}$), β^i is three-dimensional shift vector (which determines change in spatial coordinates) and η_{ij} ($i, j = 1, 2, 3$) are the components of three-dimensional hypersurfaces. The SdS spacetime in Rindler coordinates is given by (Ali and Rahman 2009)

$$ds^2 = -\alpha^2(z) dt^2 + dx^2 + dy^2 + dz^2, \quad (2.2)$$

where the directions z , y and x are analogous to the Schwarzschild coordinates r , ϕ and θ respectively. Since SdS black hole is non-rotating, the shift vector vanishes. On comparing (2.1) and (2.2), we have

$$\alpha = \alpha(z), \quad \beta = 0, \quad \eta_{ij} = 1 \quad (i = j). \quad (2.3)$$

The 3 + 1 GRMHD equations for the line element (2.2) in a Veselago medium are given by (A5)–(A9) in Appendix. The equation of state for isothermal plasma is Zhang (1989a)

$$\mu = \frac{\rho + p}{\rho_0} = \text{constant}, \quad (2.4)$$

here ρ_0 , ρ , p and μ denote rest mass density, moving mass density, pressure and specific enthalpy respectively. The specific enthalpy is constant while pressure $p \neq 0$ for the isothermal plasma. This equation shows that there is no energy exchange between plasma and magnetic field of fluid. The corresponding 3 + 1 GRMHD equations (A5)–(A9) for isothermal plasma surrounding the SdS black hole become

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{V} \times \mathbf{B}), \tag{2.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.6}$$

$$\frac{\partial(\rho + p)}{\partial t} + (\rho + p)\gamma^2 \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + (\alpha \mathbf{V} \cdot \nabla)(\rho + p) + (\rho + p)\gamma^2 V \cdot (\alpha \mathbf{V} \cdot \nabla) \mathbf{V} + (\rho + p)\nabla \cdot (\alpha \mathbf{V}) = 0, \tag{2.7}$$

$$\left\{ \left((\rho + p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \delta_{ij} + (\rho + p)\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \times \left(\frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) V^j - \left(\frac{\mathbf{B}^2}{4\pi} \delta_{ij} - \frac{1}{4\pi} B_i B_j \right) V^j_{,k} V^k + (\rho + p)\gamma^2 a_i + p_{,i} = \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B})_i \nabla \cdot (\mathbf{V} \times \mathbf{B}) - \frac{1}{8\pi\alpha^2} (\alpha \mathbf{B})^2_{,i} + \frac{1}{4\pi\alpha} (\alpha B_i)_{,j} B^j - \frac{1}{4\pi\alpha} [\mathbf{B} \times \{ \mathbf{V} \times (\nabla \times (\alpha \mathbf{V} \times \mathbf{B})) \}]_i, \tag{2.8}$$

$$\left(\frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (\rho + p)\gamma^2 - \frac{1}{\alpha} \frac{\partial p}{\partial t} + 2(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{a}) + (\rho + p) \times \gamma^2 (\nabla \cdot \mathbf{V}) - \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}) \cdot \left(\mathbf{V} \times \frac{\partial \mathbf{B}}{\partial t} \right) - \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}) \cdot \left(\mathbf{B} \times \frac{\partial \mathbf{V}}{\partial t} \right) + \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \alpha \mathbf{B}) = 0. \tag{2.9}$$

In rotating background, we assume that plasma flow is in two dimensions, i.e., in xz -plane. Therefore FIDO’s measured velocity \mathbf{V} and magnetic field \mathbf{B} turn out to be

$$\mathbf{V} = V(z)\mathbf{e}_x + u(z)\mathbf{e}_z, \quad \mathbf{B} = B[\lambda(z)\mathbf{e}_x + \mathbf{e}_z], \tag{2.10}$$

here B is an arbitrary constant. The relation between the quantities λ , u and V is given by Sharif and Sheikh (2007a, 2007b, 2008a, 2008b, 2008c)

$$V = \frac{V^F}{\alpha} + \lambda u, \tag{2.11}$$

where V^F is an integration constant. The Lorentz factor, $\gamma = \frac{1}{\sqrt{1-\mathbf{V}^2}}$ becomes

$$\gamma = \frac{1}{\sqrt{1-u^2-V^2}}. \tag{2.12}$$

When the plasma flow is perturbed due to black hole gravity, we use linear perturbation. The flow variables (mass density ρ , pressure p , velocity \mathbf{V} and magnetic field \mathbf{B}) take the form

$$\rho = \rho^0 + \delta\rho = \rho^0 + \rho\tilde{\rho}, \quad p = p^0 + \delta p = p^0 + p\tilde{p}, \\ \mathbf{V} = \mathbf{V}^0 + \delta\mathbf{V} = \mathbf{V}^0 + \mathbf{v}, \quad \mathbf{B} = \mathbf{B}^0 + \delta\mathbf{B} = \mathbf{B}^0 + \mathbf{B}\mathbf{b}, \tag{2.13}$$

where unperturbed quantities are denoted by ρ^0 , p , \mathbf{V}^0 and \mathbf{B}^0 , the linearly perturbed quantities are represented by $\delta\rho$, δp , $\delta\mathbf{V}$ and $\delta\mathbf{B}$. We introduce the following dimensionless quantities $\tilde{\rho}$, \tilde{p} , v_x , v_z , b_x and b_z for the perturbed quantities

$$\tilde{\rho} = \tilde{\rho}(t, z), \quad \tilde{p} = \tilde{p}(t, z), \\ \mathbf{v} = \delta\mathbf{V} = v_x(t, z)\mathbf{e}_x + v_z(t, z)\mathbf{e}_z, \tag{2.14} \\ \mathbf{b} = \frac{\delta\mathbf{B}}{B} = b_x(t, z)\mathbf{e}_x + b_z(t, z)\mathbf{e}_z.$$

When we insert these linear perturbations in the perfect GRMHD equations (2.5)–(2.9) along with (2.14), the component form of these equations will be Sharif and Mukhtar (2011a, 2011b)

$$\frac{1}{\alpha} \frac{\partial b_x}{\partial t} - ub_{x,z} = (ub_x - Vb_z - v_x + \lambda v_z)\nabla \ln \alpha - (v_{x,z} - \lambda v_{z,z} - \lambda'v_z + V'b_z + Vb_{z,z} - u'b_x), \tag{2.15}$$

$$\frac{1}{\alpha} \frac{\partial b_z}{\partial t} = 0, \tag{2.16}$$

$$b_{z,z} = 0, \tag{2.17}$$

$$\rho \frac{\partial \tilde{\rho}}{\partial t} + p \frac{\partial \tilde{p}}{\partial t} + (\rho + p)\gamma^2 \left(V \frac{\partial v_x}{\partial t} + u \frac{\partial v_z}{\partial t} \right) + \alpha u \rho \rho_{,z} + \alpha u p p_{,z} + \alpha(\rho + p) \{ \gamma^2 u V v_{x,z} + (1 + \gamma^2 u^2) v_{z,z} \} - \frac{1}{\gamma} (\tilde{\rho} - \tilde{p})(\alpha u \gamma p)_{,z} + \alpha(\rho + p)\gamma^2 u \{ (1 + 2\gamma^2 V^2) V' + 2\gamma^2 u V u' \} v_x - \alpha(\rho + p) \left\{ (1 - 2\gamma^2 u^2)(1 + \gamma^2 u^2) \frac{u'}{u} \right\} - 2\gamma^4 u^2 V V' v_z = 0, \tag{2.18}$$

$$\left\{ (\rho + p)\gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_z}{\partial t} + \left\{ (\rho + p)\gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} u v_{x,z} + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} u v_{z,z} - \frac{B^2}{4\pi} (1 + u^2) b_{x,z} - \frac{B^2}{4\pi\alpha} \{ \alpha'(1 + u^2) + \alpha u u' \} b_x + \gamma^2 u (\rho \tilde{\rho} + p \tilde{p}) \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \}$$

$$\begin{aligned}
& + \left[(\rho + p)\gamma^4 u \{ (1 + 4\gamma^2 V^2) uu' \right. \\
& + 4VV'(1 + \gamma^2 V^2) \} + \left. \frac{B^2 u \alpha'}{4\pi \alpha} \right] v_x \\
& + \left[(\rho + p)\gamma^2 \{ (1 + 2\gamma^2 u^2)(1 + 2\gamma^2 V^2) V' - \gamma^2 V^2 V' \right. \\
& + 2\gamma^2 (1 + 2\gamma^2 u^2) u V u' \} - \left. \frac{B^2 u}{4\pi \alpha} (\lambda \alpha)' \right] v_z = 0, \quad (2.19)
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\rho + p)\gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_z}{\partial t} \\
& + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_x}{\partial t} \\
& + \left\{ (\rho + p)\gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} u v_{z,z} \\
& + \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} u v_{x,z} + \frac{\lambda B^2}{4\pi} (1 + u^2) b_{x,z} \\
& + \frac{B^2}{4\pi \alpha} \{ \alpha' \lambda - (\alpha \lambda)' + u \lambda (u \alpha' + u' \alpha) \} b_x \\
& + (\rho \tilde{\rho} + p \tilde{p}) \gamma^2 \{ a_z + uu'(1 + \gamma^2 u^2) + \gamma^2 u^2 V V' \} \\
& + \left[(\rho + p)\gamma^4 \{ u^2 V'(1 + 4\gamma^2 V^2) \right. \\
& + 2V(a_z + uu'(1 + 2\gamma^2 u^2)) \} - \left. \lambda B^2 \frac{u \alpha'}{4\pi \alpha} \right] v_x \\
& + \left[(\rho + p)\gamma^2 \{ u'(1 + \gamma^2 u^2)(1 + 4\gamma^2 u^2) \right. \\
& + 2u\gamma^2 (a_z + (1 + 2\gamma^2 u^2) V V') \} + \left. \frac{\lambda B^2 u}{4\pi \alpha} (\alpha \lambda)' \right] v_z \\
& + (p' \tilde{p} + p \tilde{p}') = 0, \quad (2.20) \\
& \frac{1}{\alpha} \gamma^2 \rho \frac{\partial \tilde{\rho}}{\partial t} + \frac{1}{\alpha} \gamma^2 p \frac{\partial \tilde{p}}{\partial t} + \gamma^2 (\rho' + p') v_z \\
& + u \gamma^2 (\rho \tilde{\rho}_{,z} + p \tilde{p}_{,z} + \rho' \tilde{\rho} + p' \tilde{p}) - \frac{1}{\alpha} p \frac{\partial \tilde{p}}{\partial t} \\
& + 2\gamma^2 u (\rho \tilde{\rho} + p \tilde{p}) a_z + \gamma^2 u' (\rho \tilde{\rho} + p \tilde{p}) + 2(\rho + p)\gamma^4 \\
& \times (u V' + 2u V a_z + u' V) v_x + 2(\rho + p)\gamma^2 (2\gamma^2 uu' \\
& + a_z \gamma^4 + 2\gamma^2 u^2 a_z) v_z + 2(\rho + p)\gamma^4 u V v_{x,z} \\
& + (\rho + p)\gamma^2 (1 + 2\gamma^2 u^2) v_{z,z} - \frac{B^2}{4\pi \alpha} \left[(V^2 + u^2) \lambda \frac{\partial b_x}{\partial t} \right. \\
& + (V^2 + u^2) \frac{\partial b_z}{\partial t} - V(\lambda V + u) \frac{\partial b_x}{\partial t} - u(\lambda V + u) \frac{\partial b_z}{\partial t} \left. \right] \\
& + \frac{B^2}{4\pi \alpha} [(V - \lambda u) v_{x,t} + \lambda(u \lambda - V) v_{z,t}] \\
& - \frac{B^2}{4\pi} (\lambda \lambda' v_z - \lambda' v_x - \lambda' V b_z + \lambda' u b_x - V b_{x,z} + u \lambda b_{x,z}) \\
& = 0. \quad (2.21)
\end{aligned}$$

The following harmonic spacetime dependence of perturbation is assumed for the Fourier analysis,

$$\begin{aligned}
\tilde{\rho}(t, z) &= c_1 e^{-i(\omega t - kz)}, & \tilde{p}(t, z) &= c_2 e^{-i(\omega t - kz)}, \\
v_z(t, z) &= c_3 e^{-i(\omega t - kz)}, & v_x(t, z) &= c_4 e^{-i(\omega t - kz)}, \\
b_z(t, z) &= c_5 e^{-i(\omega t - kz)}, & b_x(t, z) &= c_6 e^{-i(\omega t - kz)},
\end{aligned} \quad (2.22)$$

here ω and k represent the angular frequency and z -component of the wave vector $(0, 0, k)$, respectively. The wave vector can be used to determine refractive index and the properties of plasma near the event horizon.

The ratio of speed of light when it travels from one medium to another is said to be refractive index. Frequency dependence effects in wave propagation refers to dispersion. This describes relations between wave properties like wave length, angular frequency, refractive index etc. (Das 2004). Dispersion is said to be normal if change in the refractive index with respect to angular frequency is positive, otherwise anomalous. Using (2.22) in (2.15)–(2.21), we get their Fourier analyzed form

$$\begin{aligned}
c_4 (\alpha' + i k \alpha) - c_3 \{ (\alpha \lambda)' + i k \alpha \lambda \} \\
- c_5 (\alpha V)' + c_6 \{ (\alpha u)' + i \omega + i k u \alpha \} = 0, \quad (2.23)
\end{aligned}$$

$$c_5 \left(\frac{-i \omega}{\alpha} \right) = 0, \quad (2.24)$$

$$c_5 i k = 0, \quad (2.25)$$

$$\begin{aligned}
c_1 \{ (-i \omega + i k \alpha u) \rho - p \gamma^2 \alpha u (V V' + u u') - \alpha' u p - \alpha u' p \\
- \alpha u p' \} + c_2 \{ (-i \omega + i k \alpha u) p + \alpha' u p + \alpha u' p + \alpha u p' \\
+ p \gamma^2 \alpha u (V V' + u u') \} + c_3 (\rho + p) \left[-i \omega \gamma^2 u \right. \\
+ i k \alpha (1 + \gamma^2 u^2) - \alpha \left\{ (1 - 2\gamma^2 u^2)(1 + \gamma^2 u^2) \frac{u'}{u} \right. \\
\left. - 2\gamma^4 u^2 V V' \right\} \left. \right] + c_4 (\rho + p) [\gamma^2 V (-i \omega + i k \alpha u) \\
+ \alpha \gamma^2 u \{ (1 + 2\gamma^2 V^2) V' + 2\gamma^2 u V u' \}] = 0, \quad (2.26)
\end{aligned}$$

$$\begin{aligned}
c_1 \rho \gamma^2 u \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} \\
+ c_2 p \gamma^2 u \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} \\
+ c_3 \left[- \left\{ (\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{i \omega}{\alpha} + \left\{ (\rho + p)\gamma^4 u V \right. \right. \\
\left. \left. - \frac{\lambda B^2}{4\pi} \right\} i k u + (\rho + p)\gamma^2 \{ (1 + 2\gamma^2 u^2)(1 + 2\gamma^2 V^2) \right. \\
\left. - \gamma^2 V^2 \} V' + 2\gamma^4 (\rho + p) u V u' (1 + 2\gamma^2 u^2) \right. \\
\left. - \frac{B^2 u}{4\pi \alpha} (\alpha \lambda)' \right] + c_4 \left[- \left\{ (\rho + p)\gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} \right. \\
\left. \times \frac{i \omega}{\alpha} + \left\{ (\rho + p)\gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} i k u + (\rho + p) \right. \\
\left. \times \gamma^4 u \{ (1 + 4\gamma^2 V^2) u u' + 4V V' (1 + \gamma^2 V^2) \} + \frac{B^2 u \alpha'}{4\pi \alpha} \right] \\
- c_6 \frac{B^2}{4\pi} \left\{ (1 + u^2) i k + (1 + u^2) \frac{\alpha'}{\alpha} + u u' \right\} = 0, \quad (2.27)
\end{aligned}$$

$$\begin{aligned}
 & c_1 \rho \gamma^2 \{ a_z + uu'(1 + \gamma^2 u^2) + \gamma^2 u^2 V V' \} \\
 & + c_2 [p \gamma^2 \{ a_z + uu'(1 + \gamma^2 u^2) + \gamma^2 u^2 V V' \} + p' + ikp] \\
 & + c_3 \left[- \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} \frac{i\omega}{\alpha} \right. \\
 & + \left. \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} iku + \{ (\rho + p) \gamma^2 \right. \\
 & \times \left. \{ u'(1 + \gamma^2 u^2) (1 + 4\gamma^2 u^2) \right. \\
 & + \left. 2u\gamma^2 \{ a_z + (1 + 2\gamma^2 u^2) \} V V' \} + \frac{\lambda B^2 u}{4\pi \alpha} (\alpha \lambda)' \right] \\
 & + c_4 \left[- \left\{ (\rho + p) \gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \frac{i\omega}{\alpha} + \left\{ (\rho + p) \gamma^4 u V \right. \right. \\
 & - \left. \left. \frac{\lambda B^2}{4\pi} \right\} iku + \{ (\rho + p) \gamma^4 \{ u^2 V' (1 + 4\gamma^2 V^2) \right. \\
 & + \left. 2V \left\{ (1 + 2\gamma^2 u^2) uu' + a_z - \frac{\lambda B^2 \alpha' u}{4\pi \alpha} \right\} \right] \\
 & + c_6 \left[\frac{B^2}{4\pi \alpha} \{ -(\alpha \lambda)' + \alpha' \lambda - u \lambda (u \alpha' + u' \alpha) \} \right. \\
 & + \left. \frac{\lambda B^2}{4\pi} (1 + u^2) ik \right] = 0, \tag{2.28}
 \end{aligned}$$

$$\begin{aligned}
 & c_1 \left\{ \left(\frac{-i\omega}{\alpha} \gamma^2 + iku\gamma^2 + 2u\gamma^2 a_z + \gamma^2 u' \right) \rho + u\rho' \gamma^2 \right\} \\
 & + c_2 \left\{ \left(\frac{i\omega}{\alpha} (1 - \gamma^2) + iku\gamma^2 + 2\gamma^2 u a_z + \gamma^2 u' \right) p \right. \\
 & + \left. u\gamma^2 p' \right\} + c_3 \gamma^2 \left\{ (\rho' + p') \right. \\
 & + \left. 2(2\gamma^2 uu' + a_z + 2\gamma^2 u^2 a_z) (\rho + p) \right. \\
 & + \left. (1 + 2\gamma^2 u^2) (\rho + p) ik + \frac{\lambda B^2}{4\pi \alpha} (\lambda u - V) i\omega + \alpha \lambda' \right\} \\
 & + c_4 \left[2(\rho + p) \gamma^4 \{ (uV' + 2uV a_z + u'V) + uV ik \} \right. \\
 & + \left. \frac{B^2}{4\pi \alpha} (V - u\lambda) i\omega - \alpha \lambda' \right] + c_6 \left[\frac{-B^2}{4\pi \alpha} \{ (V^2 + u^2) \lambda \right. \\
 & + \left. V(\lambda V + u) i\omega \} - \alpha \lambda' u + ik\alpha (V - u\lambda) \right] = 0. \tag{2.29}
 \end{aligned}$$

3 Rotating non-magnetized flow

For the rotating non-magnetized background of plasma flow, we substitute $B = 0 = \lambda$ and $c_5 = 0 = c_6$ in the Fourier analyzed perturbed GRMHD equations (2.26)–(2.29) (Sharif and Mukhtar 2011a, 2011b).

3.1 Numerical solutions

For the rotating non-magnetized plasma, we use the following assumptions to find out the numerical solutions

1. Specific enthalpy: $\mu = 1$,
2. Time lapse: $\alpha = \frac{z}{2r_h}$,
3. Velocity components: $u = V$, x and z -components of velocity yield $u = V = -\frac{1}{\sqrt{z^2+2}}$,
4. Stiff fluid: $\rho = p = -\frac{1}{2u}$,
5. Lorentz factor: $\gamma = \frac{1}{\sqrt{1-u^2-V^2}} = \frac{\sqrt{z^2+2}}{z}$,

where r_h is the SdS event horizon greater than that of the Schwarzschild event horizon and $r_h \approx 2M(1 + \frac{4M^2}{l^2} + \dots) \approx \zeta 0.2948$ km, $1 \leq \zeta \leq 1.5$ for a black hole mass $M \sim 1M_\odot$. The value of ζ corresponding to extremal SdS black hole is 1.5 (Ali and Rahman 2009).

We consider the region $-5 \leq z \leq 5$ for wave analysis assuming that event horizon is at $z = 0$. We take this region to explain waves near horizon only for convenience. Since the flow variables exhibit large variations in the region $-1 \leq z \leq 1$, we ignore it and solve dispersion relation for two meshes, i.e., $-5 \leq z \leq -1$ and $1 \leq z \leq 5$ (corresponding to near and far electromagnetic radiation zone). A complex dispersion relation (Das 2004) is obtained by solving the determinant of the coefficients of constants of the corresponding equations of the rotating non-magnetized plasma. The real part of the determinant yields a quartic equation in k

$$\begin{aligned}
 & A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k \\
 & + A_5(z, \omega) = 0 \tag{3.1}
 \end{aligned}$$

which gives four real roots. A cubic equation in k is obtained from the imaginary part

$$B_1(z)k^3 + B_2(z, \omega)k^2 + B_3(z, \omega)k + B_4(z, \omega) = 0 \tag{3.2}$$

which yields three real roots. The first and second root of the real part show wave propagation only in the region $-5 \leq z \leq -1$ while the third and fourth root exhibit waves in the region $1 \leq z \leq 5$. The roots of the imaginary part indicate wave propagation in both meshes $-5 \leq z \leq -1$ and $1 \leq z \leq 5$, i.e., region near the event horizon and outer end of magnetosphere respectively.

The wave vector, refractive index, its change with respect to angular frequency, group velocity and phase velocity lead to the wave properties of the SdS black hole and properties of Veselago medium. These are shown in Figs. 1–10 by using real values of k in (3.1) and (3.2).

It is given that dispersion is normal if phase velocity is greater than the group velocity, otherwise anomalous (Achenbach 1973) or equivalently dispersion is normal if change in refractive index is positive, anomalous otherwise. We see from figures that some waves move towards the event horizon and some move away from the horizon. The dispersion is normal in Figs. 5–7 and 10 while it is anomalous in the whole region of Fig. 8. The following table classifies the regions of normal and anomalous dispersion in Figs. 1–4 and 9.

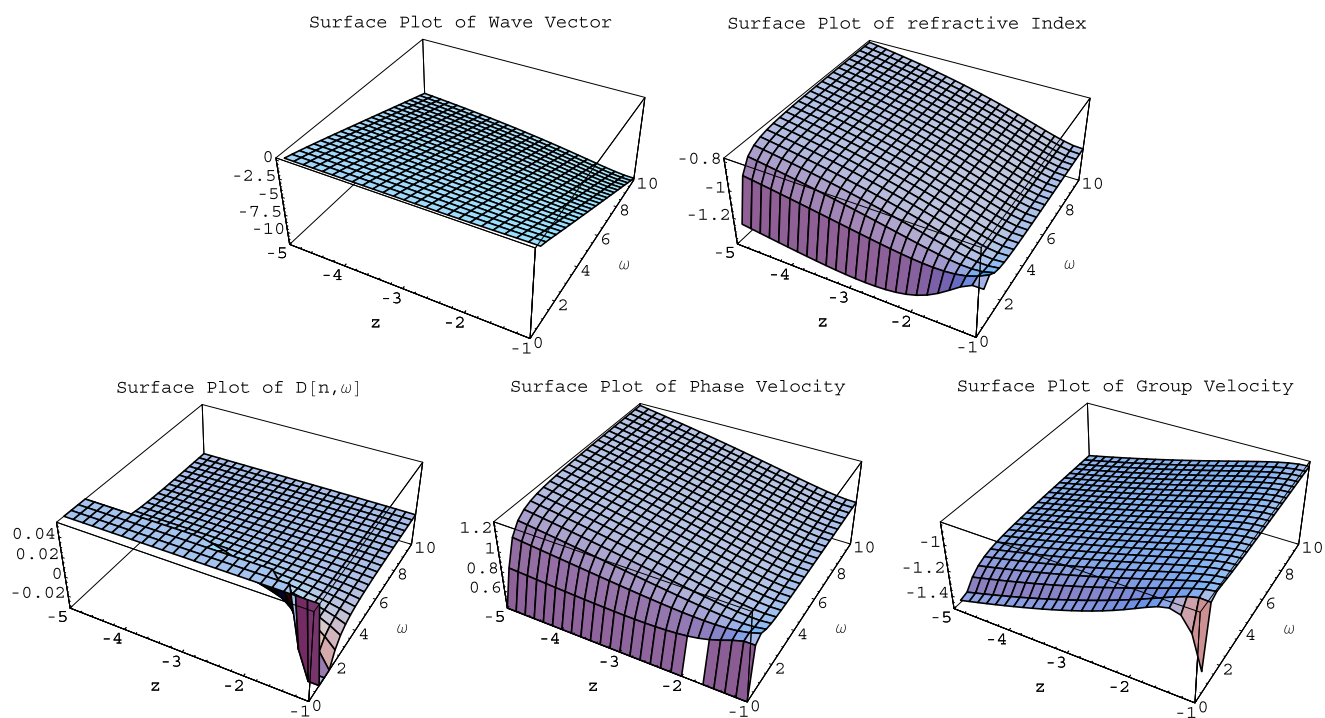


Fig. 1 Dispersion is normal and anomalous in the region

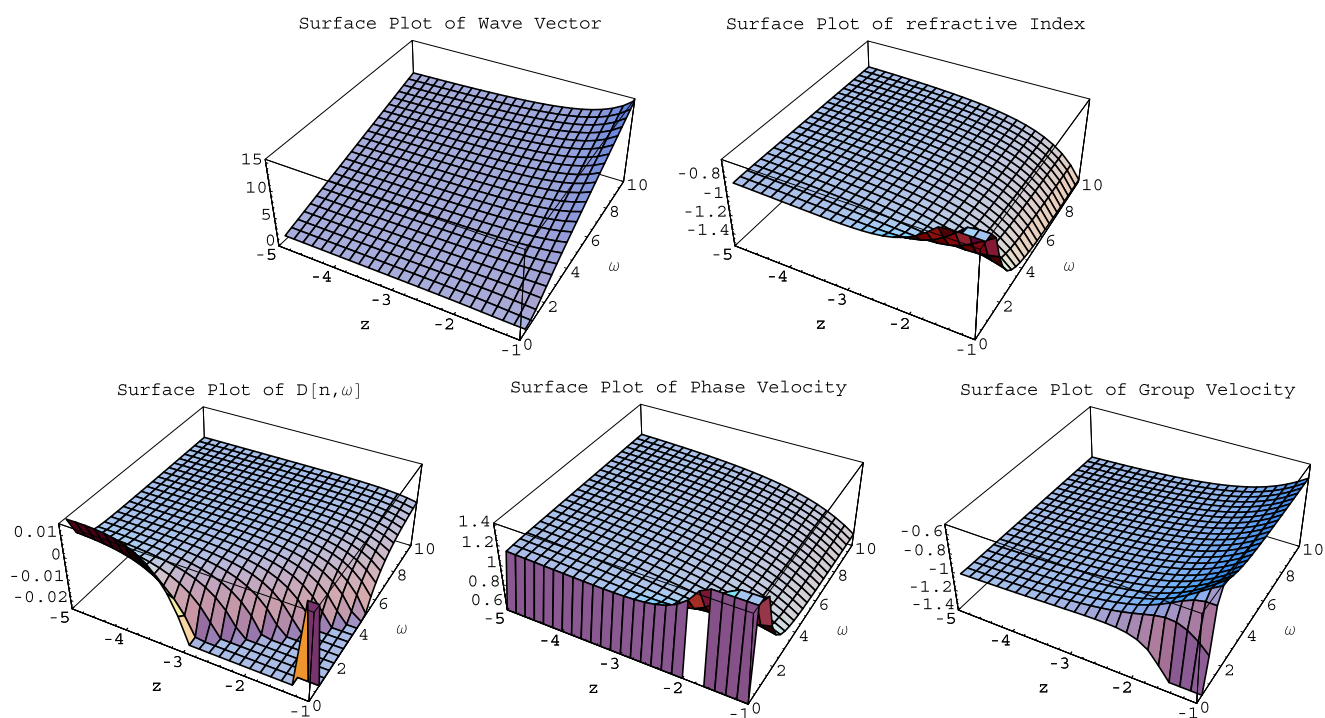


Fig. 2 Normal as well as anomalous dispersion occur at random points in the region

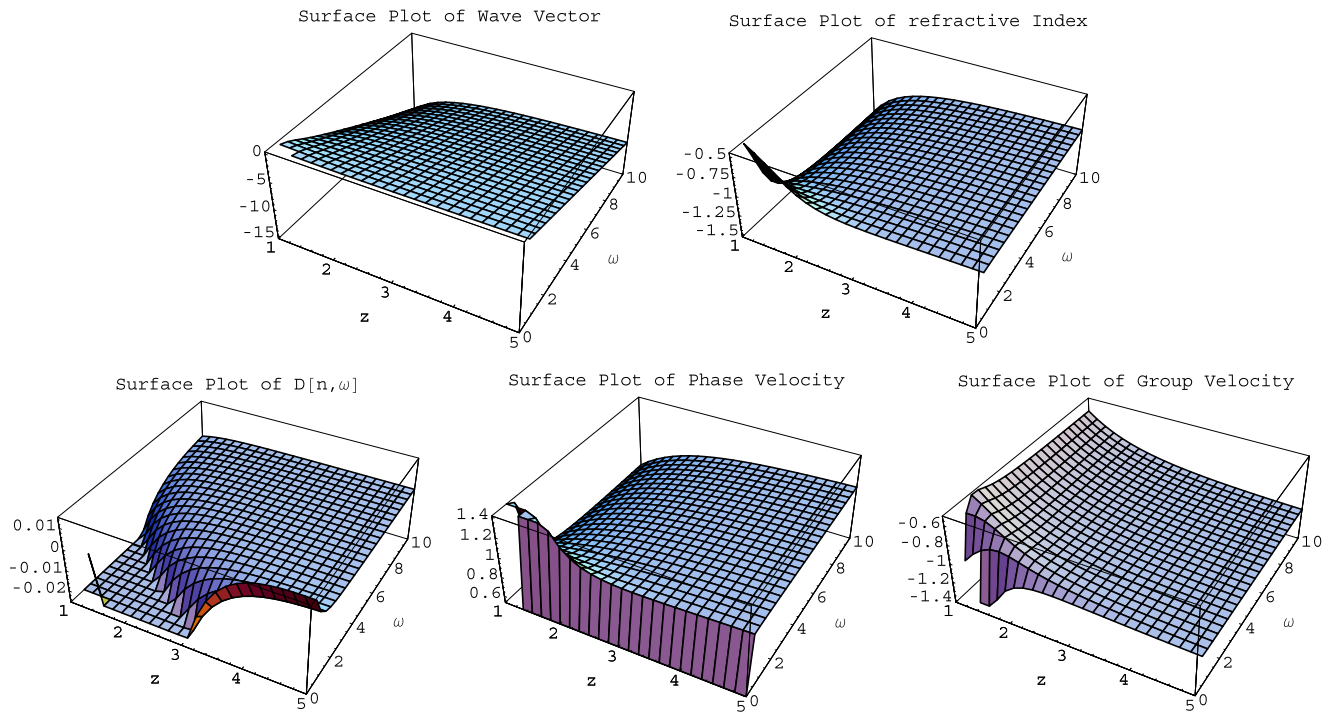


Fig. 3 Normal and anomalous dispersion of waves is observed

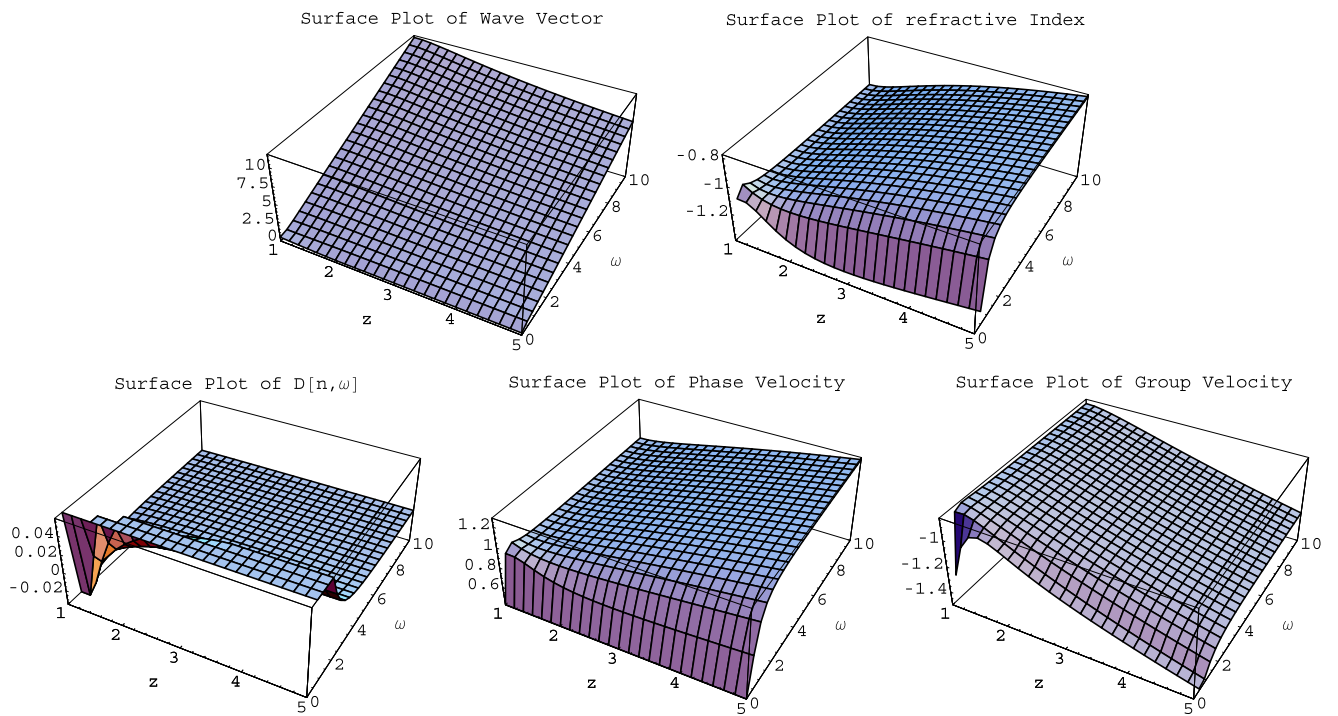


Fig. 4 Random points of normal and anomalous dispersion are found in the region

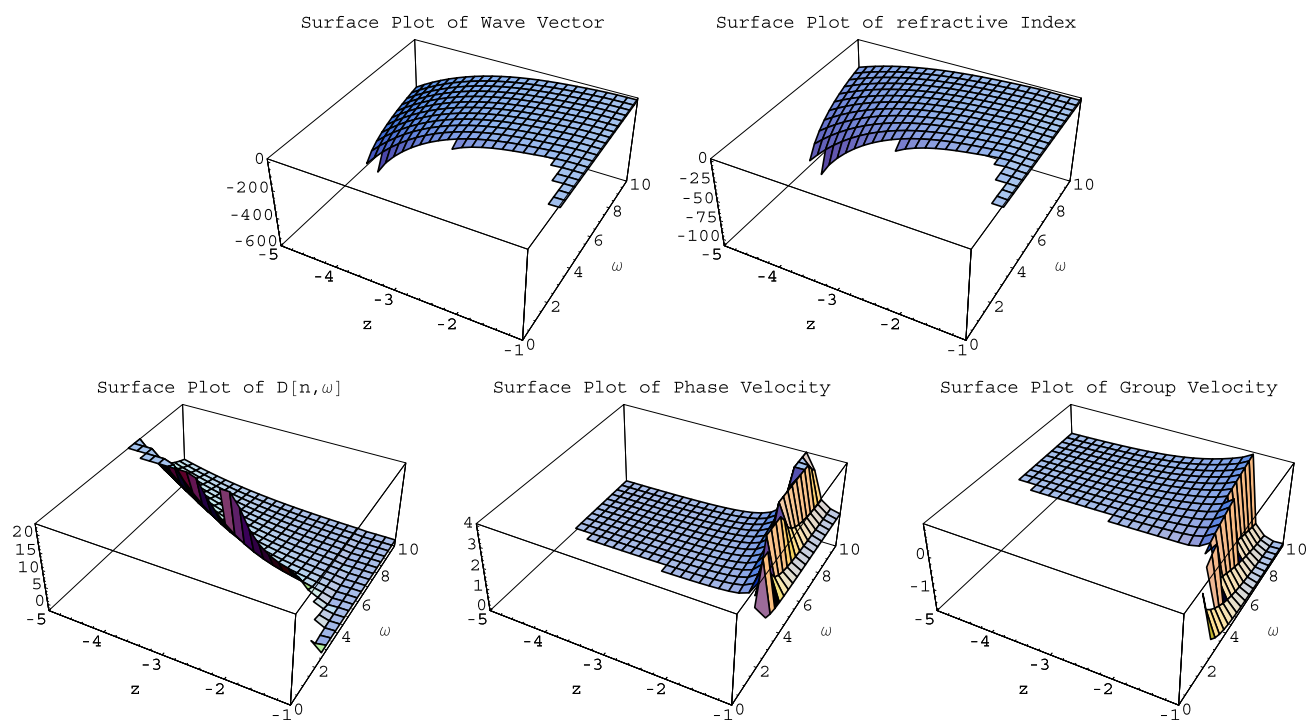


Fig. 5 Whole region admits normal dispersion

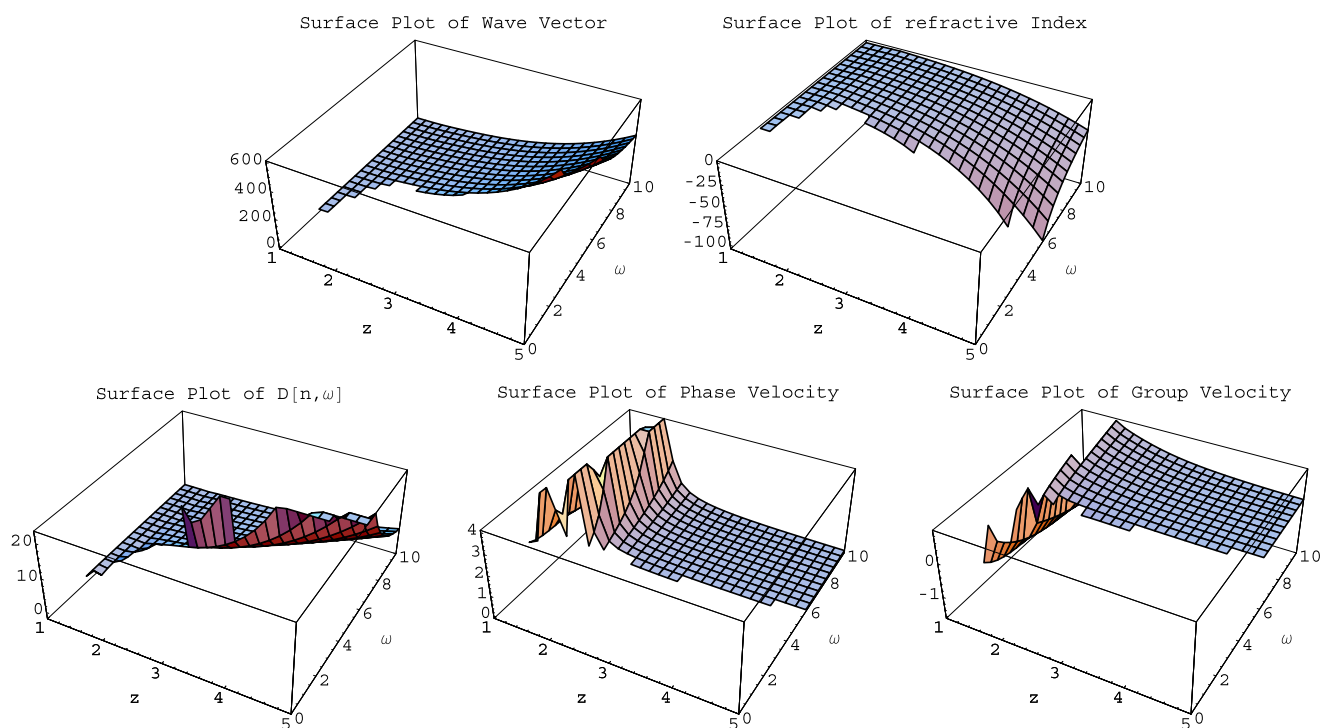


Fig. 6 Dispersion of waves is normal throughout the region

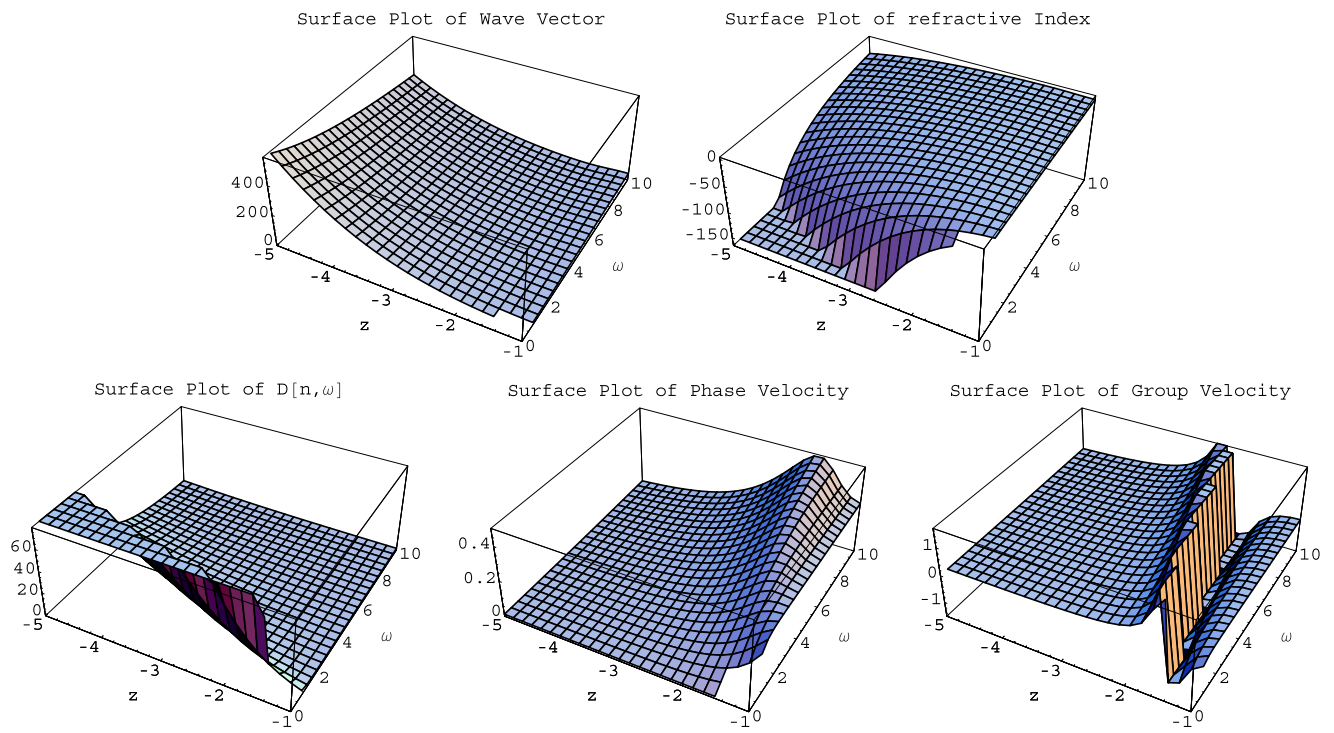


Fig. 7 Waves disperse normally in the whole region

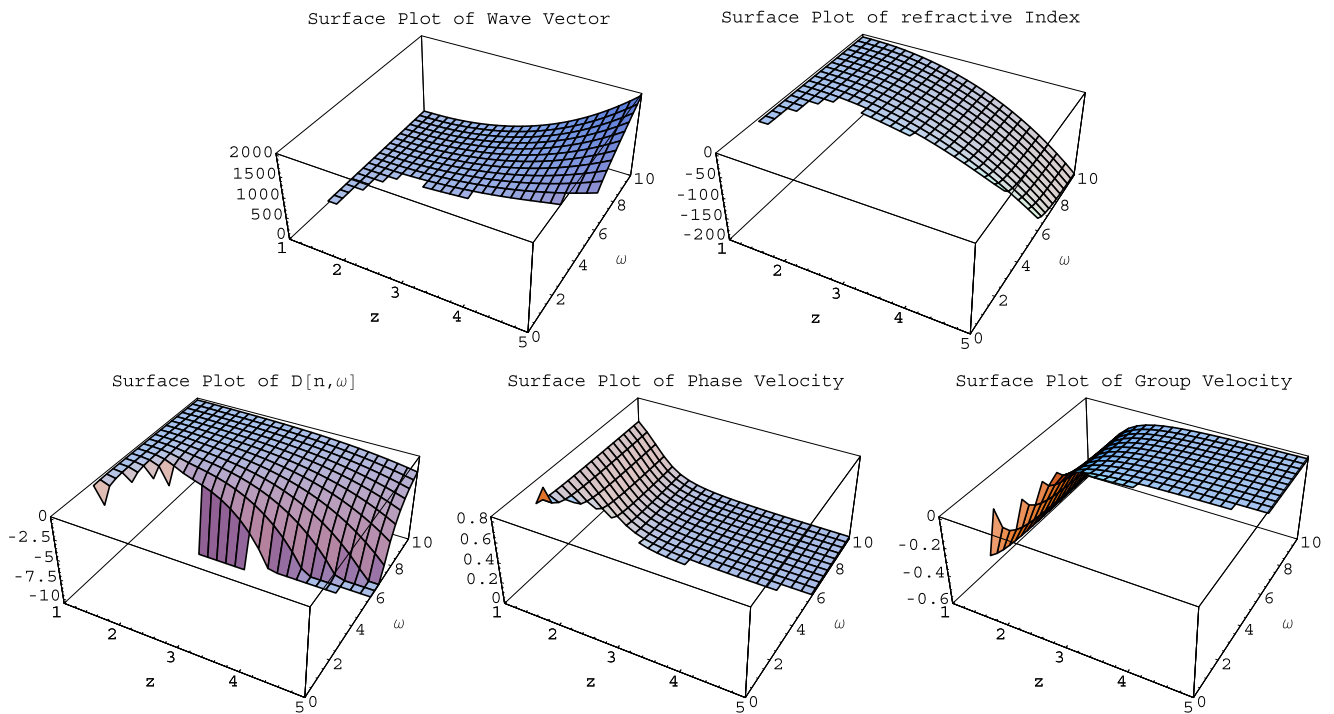


Fig. 8 Region shows anomalous dispersion

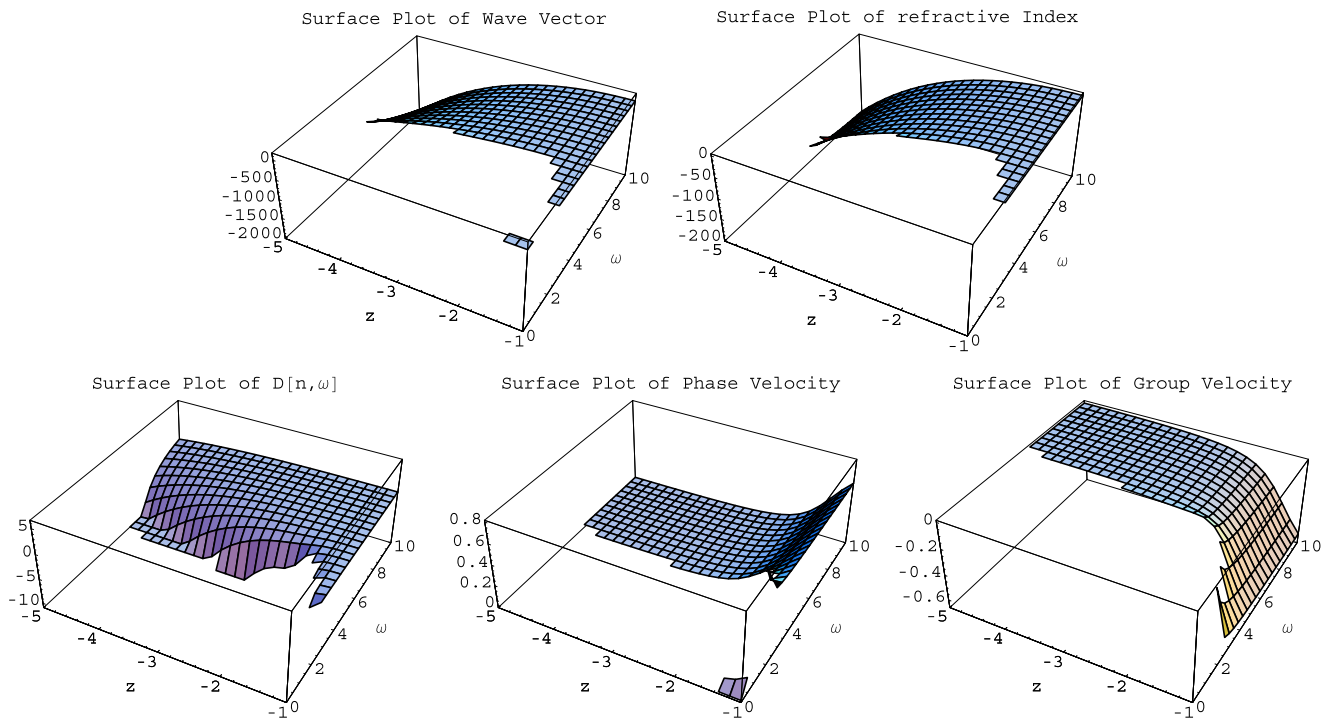


Fig. 9 Waves exhibit both normal and anomalous dispersion

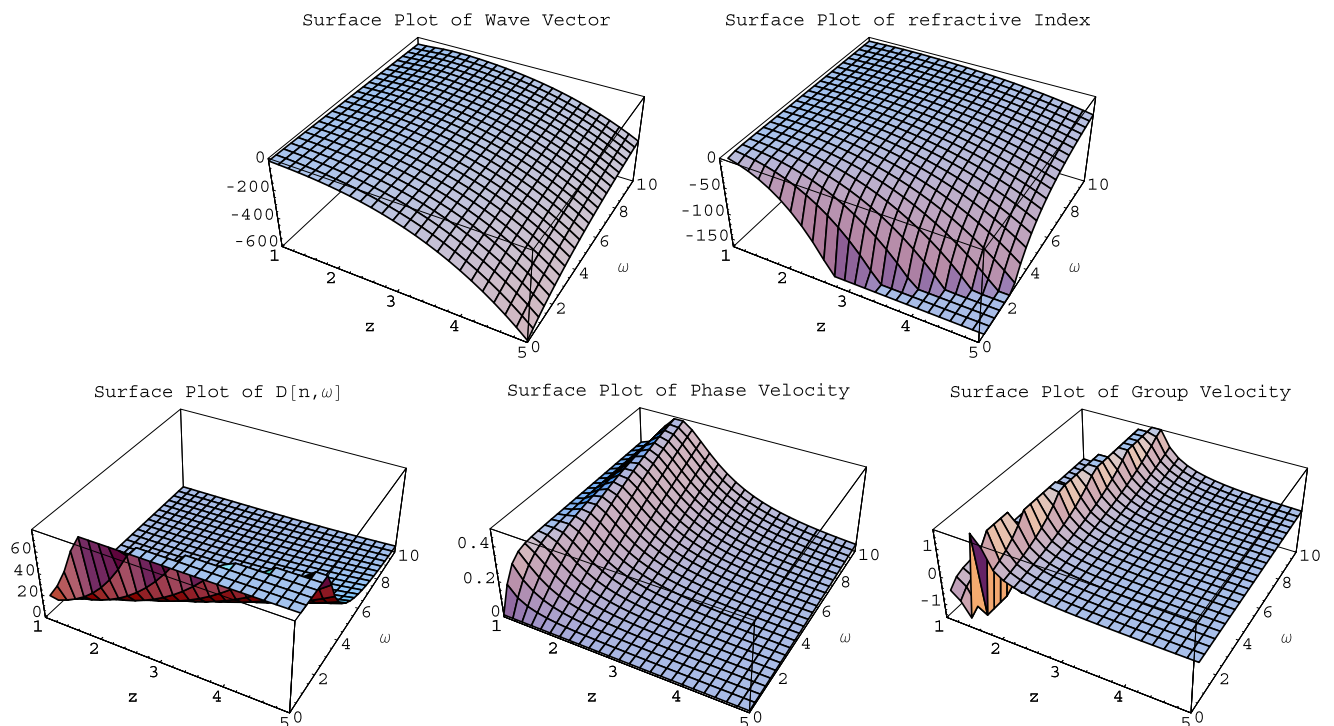


Fig. 10 Dispersion is found to be normal in the whole region

Table 1 Direction and refractive index of waves

Fig.	Direction of waves	Refractive index (n)
1	Move towards the event horizon	$n < 1$ and decreases in the region $-5 \leq z \leq -1.4, 0 \leq \omega \leq 2.6$ with the decrease in z
2	Move away from the event horizon	$n < 1$ and increases in the region $-2.94 \leq z \leq -1.1, 0 \leq \omega \leq 3.5$ with the decrease in z
3	Move towards the event horizon	$n < 1$ and increases in the region $1 \leq z \leq 1.9, 1.6 \leq \omega \leq 2.7$ with the decrease in z
4	Move outwards from the event horizon	$n < 1$ and decreases in the region $1 \leq z \leq 5, 0 \leq \omega \leq 3.8$ with the decrease in z
5	Move towards the event horizon	$n < 1$ and decreases in the region $-5 \leq z \leq -4, 5 \leq \omega \leq 8$ with the decrease in z
6	Move away from the event horizon	$n < 1$ and decreases in the region $1.8 \leq z \leq 5, 5.9 \leq \omega \leq 10$ with the decrease in z
7	Move away from the event horizon	$n < 1$ and decreases in the region $-5 \leq z \leq -2.6, 0 \leq \omega \leq 4.7$ with the decrease in z
8	Move away from the event horizon	$n < 1$ and decreases in the region $1 \leq z \leq 5, 6.3 \leq \omega \leq 10$ with the decrease in z
9	Move towards the event horizon	$n < 1$ and decreases in the region $-5 \leq z \leq -3.4, 8.1 \leq \omega \leq 10$ with the decrease in z
10	Move toward the event horizon	$n < 1$ and decreases in the region $1.4 \leq z \leq 5, 0.8 \leq \omega \leq 5.4$ with the decrease in z

Table 2 Regions of dispersion

Fig.	Normal dispersion	Anomalous dispersion
1	$-5 \leq z \leq -1.3, 0 \leq \omega \leq 10$ $-1.2 \leq z \leq -1, 0 \leq \omega \leq 1$	$-1.28 \leq z \leq -1.2, 3 \leq \omega \leq 5$ $-1.28 \leq z \leq -1.2, 7 \leq \omega \leq 10$
2	$-5 \leq z \leq -4, 0 \leq \omega \leq 10$ $-4 \leq z \leq -3, 0 \leq \omega \leq 1$ $-1.5 \leq z \leq -1, 0 \leq \omega \leq 1$	$-3 \leq z \leq -2.5, 7 \leq \omega \leq 8.5$ $-2.5 \leq z \leq -2, 7 \leq \omega \leq 8.4$ $-2 \leq z \leq -1.5, 2.5 \leq \omega \leq 3$
3	$3.9 \leq z \leq 5, 0 \leq \omega \leq 10$	$1.2 \leq z \leq 3.6, 0 \leq \omega \leq 10$
4	$1.5 \leq z \leq 5, 0 \leq \omega \leq 10$ $1 \leq z \leq 1.5, 0 \leq \omega \leq 2$	$1 \leq z \leq 1.1, 3 \leq \omega \leq 4$ $1.1 \leq z \leq 1.3, 5 \leq \omega \leq 7$
9	$-5 \leq z \leq -1, 0 \leq \omega \leq 4.5$ $-4 \leq z \leq -3, 4.7 \leq \omega \leq 5$	$-5 \leq z \leq -4, 8.2 \leq \omega \leq 10$ $-4 \leq z \leq -3, 8.5 \leq \omega \leq 10$

The results deduced from these figures can be expressed in the following Tables 1, 2.

4 Plasma flow with rotating magnetized background

Here plasma is supposed to be rotating and magnetized. The magnetic field and velocity of fluid are assumed to lie in xz -plane. The corresponding perturbed Fourier analyzed GRMHD equations, i.e., (2.23)–(2.29) are given in Sect. 2.

4.1 Numerical solutions

We take the same assumptions for the lapse function, velocity and specific enthalpy as in the previous section. Further, we assume $\frac{B^2}{4\pi} = 2$ with $u = V$ and $V^F = 1$ in (2.11) so that $\lambda = 1 + \frac{\sqrt{2+z^2}}{z}$.

Here we also consider the region $-5 \leq z \leq 5, 0 \leq \omega \leq 10$ and investigate the wave properties in meshes $-5 \leq z \leq -1$ and $1 \leq z \leq 5$. From (2.24)–(2.25), it follows that $c_5 = 0$. Consequently, we obtain dispersion relation whose real part is

$$A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k + A_5(z, \omega) = 0 \tag{4.1}$$

giving four imaginary roots. The imaginary part of the dispersion relation

$$B_1(z)k^5 + B_2(z, \omega)k^4 + B_3(z, \omega)k^3 + B_4(z, \omega)k^2 + B_5(z, \omega)k + B_6(z, \omega) = 0 \tag{4.2}$$

yields five roots of k out of which one is real and four roots are complex. The real root indicates wave propagation in both meshes, i.e., $-5 \leq z \leq -1$ and $1 \leq z \leq 5$ shown in Figs. 11, 12. This shows that waves move towards the event horizon. Also, it is obvious from figures that dispersion is normal as well as anomalous at random points.

The following Tables 3, 4 show the results obtained from these figures.

5 Summary

This paper deals with the study of isothermal plasma wave properties in magnetosphere of SdS black hole in a Veselago medium. The ADM 3 + 1 formalism has been used to formulate the GRMHD equations for this unusual medium.

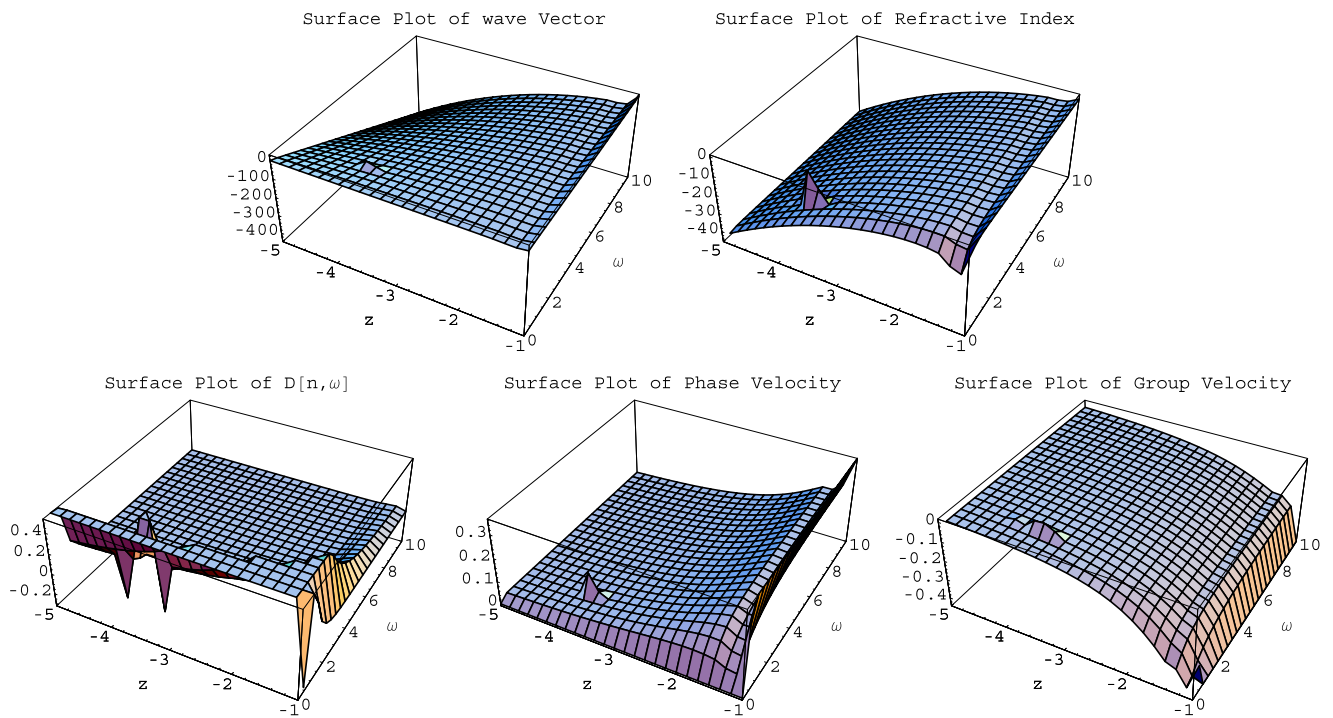


Fig. 11 Normal and anomalous dispersion at random points

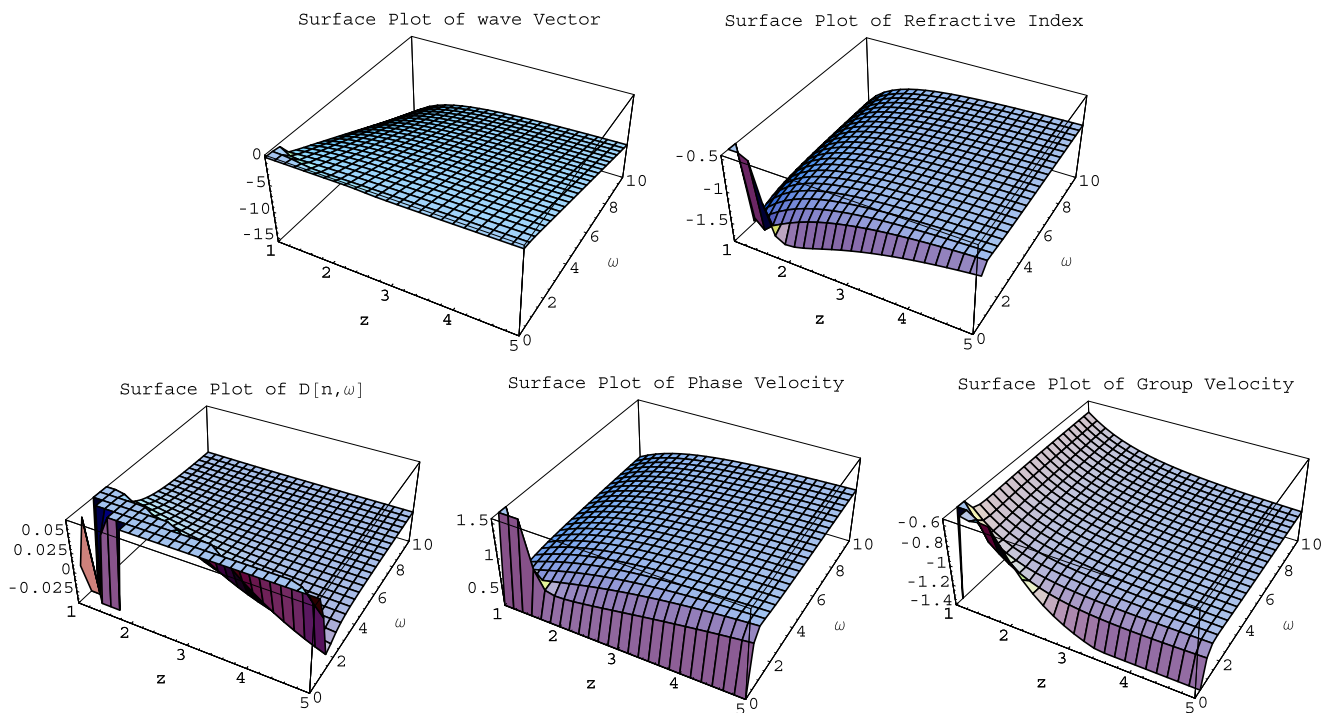


Fig. 12 Dispersion is normal as well as anomalous at random points

Table 3 Direction and refractive index of waves

Fig.	Direction of waves	Refractive index (n)
11	Move towards the event horizon	$n < 1$ and decreases in the region $-5 \leq z \leq -2.1, 0 \leq \omega \leq 10$ with the decrease in z
12	Move towards the event horizon	$n < 1$ and increases in the region $1 \leq z \leq 1.8, 1.5 \leq \omega \leq 4$ with the decrease in z

Table 4 Regions of dispersion

Fig.	Normal dispersion	Anomalous dispersion
11	$-5 \leq z \leq -4.9, 1 \leq \omega \leq 10$ $-4 \leq z \leq -3, 2.5 \leq \omega \leq 10$ $-3 \leq z \leq -1, 2.8 \leq \omega \leq 10$	$-4 \leq z \leq -3.6, 1 \leq \omega \leq 1.5$ $-3.7 \leq z \leq -3.5, 1.9 \leq \omega \leq 2.1$ $-3.5 \leq z \leq -3.35, 1.8 \leq \omega \leq 2.2$
12	$1 \leq z \leq 2, 2 \leq \omega \leq 10$ $2 \leq z \leq 4.5, 5 \leq \omega \leq 10$ $4.1 \leq z \leq 4.5, 5.1 \leq \omega \leq 10$	$1 \leq z \leq 2, 0.8 \leq \omega \leq 1.1$ $4 \leq z \leq 4.5, 1.4 \leq \omega \leq 4$ $4 \leq z \leq 4.5, 1.4 \leq \omega \leq 4.5$

We have applied linear perturbations to the GRMHD equations and have obtained their component form with the assumption that plasma flows in two dimensions. Finally, we have obtained dispersion relations for the rotating (non-magnetized and magnetized) background.

For the rotating non-magnetized background, waves move towards the event horizon shown in Figs. 1, 3, 5, 9 and 10 while waves are directed away from the event horizon in Figs. 2, 4, 6, 7 and 8. The dispersion is found to be normal as well as anomalous at random points in Figs. 1, 2, 3, 4 and 9. The Figs. 5, 6, 7 and 10 show normal dispersion while 8 admits anomalous dispersion in the whole region. The Figs. 11 and 12 indicate that waves are directed towards the event horizon for rotating magnetized plasma. It is clear from these figures that region admits normal and anomalous dispersion at random points.

We know that the refractive index is always greater than one in the usual medium, while it is less than one for the Veselago medium. Here we have found that the refractive index is less than one and increases in small regions. The phase velocity is greater than group velocity for both non-magnetized and magnetized backgrounds. These are prominent aspects of the Veselago medium which confirms the presence of this unusual medium for both rotating (non-magnetized and magnetized) plasma in SdS black hole.

It is interesting to mention here that in a recent work (Sharif and Mukhtar 2011a, 2011b) for isothermal plasma on Schwarzschild black hole, there does not exist waves for the rotating magnetized plasma. However, we have seen wave propagation in SdS black hole for this case. Here waves admit normal dispersion at most of points while for the Schwarzschild black hole, most of the waves disperse anomalously. Thus it can be concluded that more information can be extracted from magnetosphere by inclusion of

the de-Sitter patch in the Schwarzschild spacetime. It would be interesting to extend this analysis for hot plasma which is in progress.

Appendix

The Maxwell equations, the 3 + 1 GRMHD equations for the SdS spacetime are given in this appendix. The Maxwell equations for such a medium are

$$\nabla \cdot \mathbf{B} = 0, \tag{A1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{A2}$$

$$\nabla \cdot \mathbf{E} = -\frac{\rho_e}{\epsilon}, \tag{A3}$$

$$\nabla \times \mathbf{B} = -\mu \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} = 0. \tag{A4}$$

The GRMHD equations for the SdS spacetime in Rindler coordinates turn out to be (Sharif and Mukhtar 2011a, 2011b)

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{V} \times \mathbf{B}), \tag{A5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{A6}$$

$$\frac{\partial \rho_0}{\partial t} + (\alpha \mathbf{V} \cdot \nabla) \rho_0 + \rho_0 \gamma^2 \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + \rho_0 \gamma^2 \mathbf{V} \cdot (\alpha \mathbf{V} \cdot \nabla) \mathbf{V} + \rho_0 \nabla \cdot (\alpha \mathbf{V}) = 0, \tag{A7}$$

$$\left\{ \left(\rho_0 \mu \gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \delta_{ij} + \rho_0 \mu \gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \times \left(\frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) V^j - \left(\frac{\mathbf{B}^2}{4\pi} \delta_{ij} - \frac{1}{4\pi} B_i B_j \right) V^{j,k} V^k + \rho_0 \gamma^2 V_i \left\{ \frac{1}{\alpha} \frac{\partial \mu}{\partial t} + (\mathbf{V} \cdot \nabla) \mu \right\} = -\rho_0 \mu \gamma^2 a_i - p_{,i} + \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B})_i \nabla \cdot (\mathbf{V} \times \mathbf{B}) - \frac{1}{8\pi \alpha^2} (\alpha \mathbf{B})^2_{,i} + \frac{1}{4\pi \alpha} (\alpha B_i)_{,j} B^j - \frac{1}{4\pi \alpha} [\mathbf{B} \times \{ \mathbf{V} \times (\nabla \times (\alpha \mathbf{V} \times \mathbf{B})) \}]_i, \tag{A8}$$

$$\left(\frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (\mu \rho_0 \gamma^2) - \frac{1}{\alpha} \frac{\partial p}{\partial t} + 2\mu \rho_0 \gamma^2 (\mathbf{V} \cdot \mathbf{a}) + \mu \rho_0 \gamma^2 (\nabla \cdot \mathbf{V}) - \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B}) \cdot \left(\mathbf{V} \times \frac{1}{\alpha} \frac{\partial \mathbf{B}}{\partial t} \right) - \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B}) \cdot \left(\mathbf{B} \times \frac{1}{\alpha} \frac{\partial \mathbf{V}}{\partial t} \right) + \frac{1}{4\pi \alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\nabla \times \alpha \mathbf{B}) = 0. \tag{A9}$$

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