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Ideal solutions for hesitant fuzzy soft sets

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Abstract. Dealing with uncertainty is a difficult task and different tools have been proposed in the literature to handle it. Hesitant fuzzy sets are highly useful in resolving situations where people hesitate when providing their preferences. In this paper, the concept of a hesitant fuzzy soft set is modified to manage the situations in which experts assess an alternative according to finite criteria in all possible values. Next a distance measure is defined between any two elements of hesitant fuzzy soft set. Technique for order preference by similarity to ideal solution is also proposed in hesitant fuzzy soft set. An example is constructed for ranking of alternatives.

Keywords: Hesitant fuzzy soft set, TOPSIS, multiple attributes group decision making

1. Introduction

Decision makers have to face many complicated decision making problems in different disciplines of real life, like that economics, environment, management, engineering and social science. These problems are quite difficult to model because of uncertainty and vagueness. The multi-criteria decision making provides an effective framework for comparison based on the evaluation of multiple conflicting criteria. Decision process of selecting a suitable alternative has to take many factors into considerations; for instance, risks, needs, benefits, etc. Representation of human preference is not suitably possible with exact numeric values for real world decision problems because there are various types of uncertainties involve in these problems. To handle uncertainty and vagueness, fuzzy set theory, probability theory and rough set theory are to be used as mathematical tool. Bellman and Zadeh [4] used the concept of fuzzy set theory in decision making for the solution of vagueness and imprecision in human preferences. Torra [16] first introduced the idea of hesitant fuzzy set (HFS) theory to manage those situations where membership degree of an element in a set is a finite set of all possible values. HFS theory can reflect the hesitancy in stating their preferences by decision makers over objects as compared to the ordinary fuzzy set theory and its different extensions. Wang et al. [18] proposed the idea of hesitant fuzzy soft set (HFSS) for the parametrized description of objects with HFS. They also applied HFSS in multi criteria decision making problem.

There are several decision making techniques available in literature. One of commonly used approaches in multiple criteria decision making problems is the technique for order of preference by similarity to ideal solution (TOPSIS). TOPSIS is a very powerful and simple technique for the choosing the best alternative. It can also be used for the ranking of alternatives and has been a subject of great interest to researchers [1, 5, 6, 10, 21]. Hwang and Yoon [9] developed TOPSIS for multi-attribute/multi-criteria decision making (MADM/MCDM) problems. Kim et al. [11] and Shih et al. [15] addressed four advantages of TOPSIS: first is that a sound logic represents the rationale of human choice; secondly a scalar value considers the best and

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worst alternative simultaneously; third advantage is that it has a simple computation process and can be easily programmed and the last but not the least advantage is that it has ability of the performance measures of all alternatives on attributes to be visualized on a polyhedron, at least for any two dimensions. Fuzzy numbers are used to obtain a fuzzy TOPSIS [3, 7, 13]. Fuzzy TOPSIS has also been used and developed by several other authors [2, 8, 12, 14, 17]. In this paper first we modify the definition of HFSS and then extend fuzzy TOPSIS for HFSS under the opinion of finite decision makers. In this new proposed method TOPSIS and HFSS is first time used simultaneously.

This contribution is set out as follows**:** Section 2 introduces fuzzy set, some basic concepts of hesitant fuzzy set (HFS) and hesitant fuzzy element (HFE). In Section 3, we propose a modified definition of hesitant fuzzy soft set (HFSS) and also gave a notion of distance between any two hesitant fuzzy soft elements (HFSE's). In Section 4, TOPSIS is constructed for HFSS. Then in Section 5, an example is given for the ranking of alternatives to see the feasibility of the proposed TOPSIS method. Conclusion is given in the last section.

2. Preliminaries

First we review some basic concepts, necessary to understand our proposal.

Let X be a universal set. A fuzzy set B in X is a mapping from X to [0, 1]. The value $B(x)$ is the degree of membership of x in B.

A hesitant fuzzy set (HFS) is defined by Torra [16] in terms of a function that returns a set of membership values for each $x \in X$. A hesitant fuzzy set on X is a function h that when applied to X returns a subset of [0, 1], which can be represented as the following mathematical symbol:

$$HF = \{(x, h(x))|x \in X\},$$

where $h(x)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set HF . For convenience, Xia and Xu [19] named $h(x)$ a hesitant fuzzy element (HFE).

A typical hesitant fuzzy set is a fuzzy set where $h(x)$ is a finite subset of [0, 1]. Examples of hesitant fuzzy sets are given below where $h(x)$ represents the possible membership values of the set at x.

It is noted that the number of values in different HFEs may be different, let $l_{h(x)}$ be the number of values in $h(x)$. In case values in an HFE are out of order; we can arrange them in such a order, that an HFE h, let $\sigma : (1, 2, ..., n) \to (1, 2, ..., n)$ be a permutation satisfying $h_{\sigma(i)} \leq h_{\sigma(i+1)}$, $i = 1, 2, ..., l_h - 1$. Xu and Xia [20] proposed that two HFEs h_1 and h_2 have the same length l and $h_{1_{\sigma(i)}} = h_{2_{\sigma(i)}}$ if and only if $h_1 = h_2$, for $i = 1, 2, ..., l$.

3. Hesitant fuzzy soft set

Wang et al. [18] proposed the idea of hesitant fuzzy soft set from the subset of parameters' set to set of all hesitant fuzzy sets of universal set. Here we propose a new way for the description of hesitant fuzzy soft set.

Let X be a universe of discourse, $E = \{e_1, e_2, ..., e_n\}$ be a set of parameters and $H(E)$ be the set of all hesitant fuzzy sets in E. $F(X)$ is called a hesitant fuzzy soft set (HFSS), where F is a mapping given by $F : X \to H(E)$. $F(x)$ is called a hesitant fuzzy soft element (HFSE) and $F(x)/e$ is a HFE.

Example 3.1. Suppose that $X = \{h_1, h_2, h_3\}$ is a set of houses and $E = \{e_1 =$cheap, $e_2 =$beautiful, $e_3 =$size, $e_4 =$location$\}$ is a set of parameters. In this case, HFSS can describe the characteristics of the house under the hesitant fuzzy information and it is shown in Table 1.

$F(x) = F(y)$ if and only if $F(x)/e = F(y)/e$ for all $e \in E$.

HFSS theory is a generalization of HFS theory. If set of parameters is singleton set then every HFSS on X is also HFS on X. Already defined union and intersection operations for HFSS in [18] are not the generalization of these operations for the case of HFS. Here we define these basic operation which are also satisfying the union and intersection of HFSS for the singleton set of parameter just like HFS.

Let $F(X) = \{F(x)/e \,|\, x \in X \text{ and } e \in E\}$ and $G(X) = \{G(x)/e \mid x \in X \text{ and } e \in E\}$ be two HFSS, then we define their union, intersection and complement as follow:

Union: $F(X) \cup G(X) = \{J(x)|J(x) = \{\max(F(x)/e, G(x)/e) \mid e \in E\}$ and $x \in X\}$ where $\max(F(x)/e, G(x)/e) = \{a \mid \min(\max(F(x)/e), \max(G(x)/e)) \leq a \leq \max(\max(F(x)/e), \max(G(x)/e)),\ a \in F(x)/e$ and $a \in G(x)/e\}$.

Table 1

HFSS $F(X)$

	e_1	e_2	e_3	e_4
h_1	{0.5, 0.7, 0.8}	{0.7, 0.1}	{0.1, 0.5}	{0.2, 0.65}
h_2	{0.1, 0.4}	{0.3, 0.8, 1}	{0.2, 0.5}	{0.1, 0.7}
h_3	{0.1, 0.7}	{0.3, 0.7}	{0.4, 0.9}	{0.1, 0.9}

Table 2
HFSS $F_1(X)$

	e_1	e_2	e_3	e_4
c_1	$\{0.5, 0.6, 0.7\}$	$\{0.7, 0.8\}$	$\{0.2, 0.4\}$	$\{0.2, 0.35\}$
c_2	$\{0.2, 0.3\}$	$\{0.6, 0.7, 0.9\}$	$\{0.6, 0.7\}$	$\{0.45, 0.55\}$
c_3	$\{0.6, 0.8\}$	$\{0.4, 0.55\}$	$\{0.8, 0.95\}$	$\{0.2\}$

Table 3
HFSS $F_2(X)$

	e_1	e_2	e_3	e_4
c_1	$\{0.4, 0.3\}$	$\{0.6, 0.8\}$	$\{0.2\}$	$\{0.5, 0.6\}$
c_2	$\{0.1\}$	$\{0.2, 0.4\}$	$\{0.6, 0.7, 0.8\}$	$\{0.8, 1\}$
c_3	$\{0.2, 0.3\}$	$\{0, 0.3, 0.4\}$	$\{0.3, 0.5\}$	$\{0.3\}$

Table 4
HFSS $F_1(X) \cup F_2(X)$

	e_1	e_2	e_3	e_4
c_1	$\{0.4, 0.5, 0.6, 0.7\}$	$\{0.7, 0.8\}$	$\{0.2, 0.4\}$	$\{0.2, 0.35\}$
c_2	$\{0.1, 0.2, 0.3\}$	$\{0.6, 0.7, 0.9\}$	$\{0.6, 0.7\}$	$\{0.45, 0.55\}$
c_3	$\{0.3, 0.6, 0.8\}$	$\{0.4, 0.55\}$	$\{0.8, 0.95\}$	$\{0.2\}$

Table 5
HFSS $F_1(X)$

	e_1	e_2	e_3	e_4
l_1	$\{0.8, 0.7\}$	$\{0.6, 0.7, 0.9\}$	$\{0.6, 0.7\}$	$\{0.45, 0.55\}$
l_2	$\{0.25, 0.2\}$	$\{0.15, 0.23\}$	$\{0.2, 0.25\}$	$\{0.4, 0.6, 0.7\}$
l_3	$\{0.1, 0\}$	$\{0.12, 0.35\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Example 3.2. Suppose that $X = \{c_1, c_2, c_3\}$ is a set of cars and $E = \{e_1 = \text{cheap}, e_2 = \text{beautiful}, e_3 = \text{comfortable}, e_4 = \text{speedy}\}$ is a set of parameters. In this case, HFSSs $F_1(X)$ and $F_2(X)$ can describe the characteristics of the cars under the hesitant fuzzy information and these sets are shown in Tables 2 and 3, respectively. Union of these HFSSs is given in Table 4.

Intersection: $F(X) \cap G(X) = \{J(x) | J(x) = \{\min (F(x)/e, G(x)/e) \mid e \in E\}$ and $x \in X\}$ where $\min(F(x)/e, G(x)/e) = \{a \mid \min(\min(F(x)/e), \min(G(x)/e)) \leq a \leq$
$\max(\min(F(x)/e), \min(G(x)/e)), a \in F(x)/e$ and $a \in G(x)/e\}$.

Example 3.3. Suppose that $X = \{l_1, l_2, l_3\}$ is a set of laptops and $E = \{e_1 = \text{cheap}, e_2 = \text{beautiful}, e_3 = \text{hard drive}, e_4 = \text{processor}\}$ is a set of parameters. In this case, HFSSs $F_1(X)$ and $F_2(X)$ can describe the characteristics of the laptops under the hesitant fuzzy information and these sets are shown in Tables 5 and 6, respectively. Intersection of these HFSSs is given in Table 7.

Table 6
HFSS $F_2(X)$

	e_1	e_2	e_3	e_4
l_2	$\{0.2, 0.1, 0\}$	$\{0.6, 0.7\}$	$\{0.2\}$	$\{0.8, 0.9\}$
l_3	$\{0.6, 0.5\}$	$\{0.7\}$	$\{0.3, 0.5\}$	$\{0.9, 1\}$

Table 7
HFSS $F_1(X) \cap F_2(X)$

	e_1	e_2	e_3	e_4
l_1	$\{0.9, 0.8, 0.7\}$	$\{0.6, 0.7, 0.9\}$	$\{0.6, 0.7\}$	$\{0.45, 0.55\}$
l_2	$\{0.2, 0.1, 0\}$	$\{0.15, 0.23\}$	$\{0.2, 0.25\}$	$\{0.4, 0.6, 0.7\}$
l_3	$\{0.5, 0.1, 0\}$	$\{0.12, 0.35\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Table 8
HFSS $F(X)$

	e_1	e_2	e_3
b_1	$\{0.6, 0.8\}$	$\{0.8, 0.95\}$	$\{0.2\}$
b_2	$\{0.9, 1\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Table 9
HFSS $(F(X))^c$

	e_1	e_2	e_3
b_1	$\{0.4, 0.2\}$	$\{0.8, 0.95\}$	$\{0.2\}$
b_2	$\{0.1, 0\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Complement: $(F(X))^c = \{(F(x))^c \mid (F(x))^c = \{(F(x)/e)^c \mid e \in E\}$ and $x \in X\}$ where $(F(x)/e)^c = \{1 - a \mid a \in F(x)/e\}$.

Example 3.4. Suppose that $X = \{b_1, b_2\}$ is a set of bikes and $E = \{e_1 = \text{cheap}, e_2 = \text{resale}, e_3 = \text{warranty}\}$ is a set of parameters. In this case, HFSS $F(X)$ can describe the characteristics of the bikes under the hesitant fuzzy information and it is shown in Table 8. Complement of this HFSS $(F(X))^c$ is given in Table 9.

Theorem 3.5. *(De Morgan's laws of hesitant fuzzy soft sets). Let $F(X)$ and $G(X)$ be two hesitant fuzzy soft sets over X; then we have*

- **a.** $(F(X) \cup G(X))^c = (F(X))^c \cap (G(X))^c$,
- **b.** $(F(X) \cap G(X))^c = (F(X))^c \cup (G(X))^c$.

Proof.

- **a.** It is enough to show that $(F(x)/e \cup G(x)/e)^c = (F(x)/e)^c \cap (G(X)/e)^c$ for any arbitrary $x \in X$ and $e \in E$.

Let $a \in F(x)/e \cup G(x)/e$ which implies that $\min(\max(F(x)/e), \max(G(x)/e)) \leq a \leq$

$\max(\max(F(x)/e), \max(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

It can be rewritten as $1-(\min(\max(F(x)/e), \max(G(x)/e))) \geq 1-a \geq 1-(\max(\max(F(x)/e), \max(G(x)/e)))$, $a \in F(x)/e$ and $a \in G(x)/e$.

It implies that

$\max(1-\max(F(x)/e), 1-\max(G(x)/e)) \geq 1-a \geq \min(1-\max(F(x)/e), 1-\max(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

Thus

$\max(\min((F(x)/e)^c), \min((G(x)/e)^c)) \geq 1-a \geq \min(\min((F(x)/e)^c), \min((G(x)/e)^c)$, $1-a \in (F(x)/e)^c$ and $1-a \in (G(x)/e)^c$.

It further implies that $1-a \in (F(x)/e)^c \cap (G(X)/e)^c$.

Therefore

$$(F(x)/e \cup G(x)/e)^c \subseteq (F(x)/e)^c \cap (G(X)/e)^c. \tag{1}$$

Conversely;

Let $1-a \in (F(x)/e)^c \cap (G(X)/e)^c$ it implies that $\max(\min((F(x)/e)^c), \min((G(x)/e)^c)) \geq 1-a \geq \min(\min((F(x)/e)^c), \min((G(x)/e)^c)$, $1-a \in (F(x)/e)^c$ and $1-a \in (G(x)/e)^c$.

It further implies that

$\max(1-\max(F(x)/e), 1-\max(G(x)/e)) \geq 1-a \geq \min(1-\max(F(x)/e), 1-\max(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

Therefore

$1-(\min(\max(F(x)/e), \max(G(x)/e))) \geq 1-a \geq 1-(\max(\max(F(x)/e), \max(G(x)/e)))$, $a \in F(x)/e$ and $a \in G(x)/e$.

Hence

$\min(\max(F(x)/e), \max(G(x)/e)) \leq a \leq$

$\max(\max(F(x)/e), \max(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

This shows that $a \in F(x)/e \cup G(x)/e$ which further implies that $1-a \in (F(x)/e \cup G(x)/e)^c$.

So

$$(F(x)/e)^c \cap (G(X)/e)^c \subseteq (F(x)/e \cup G(x)/e)^c \tag{2}$$

Equation. 1 and 2, imply that

$$(F(x)/e \cup G(x)/e)^c = (F(x)/e)^c \cap (G(X)/e)^c.$$

b. It is enough to show that $(F(x)/e \cap G(x)/e)^c = (F(x)/e)^c \cup (G(X)/e)^c$ for any arbitrary $x \in X$ and $e \in E$.

Let $a \in F(x)/e \cap G(x)/e$ which implies that $\min(\min(F(x)/e), \min(G(x)/e)) \leq a \leq$

$\max(\min(F(x)/e), \min(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

Thus

$1-(\min(\min(F(x)/e), \min(G(x)/e))) \geq 1-a \geq 1-(\max(\min(F(x)/e), \min(G(x)/e)))$, $a \in F(x)/e$ and $a \in G(x)/e$.

It implies that

$\max(1-\min(F(x)/e), 1-\min(G(x)/e)) \geq 1-a \geq \min(1-\min(F(x)/e), 1-\min(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

Therefore

$\max(\max((F(x)/e)^c), \max((G(x)/e)^c)) \geq 1-a \geq \min(\max((F(x)/e)^c), \max((G(x)/e)^c)$, $1-a \in (F(x)/e)^c$ and $1-a \in (G(x)/e)^c$.

It implies that $1-a \in (F(x)/e)^c \cup (G(X)/e)^c$.

So

$$(F(x)/e \cap G(x)/e)^c \subseteq (F(x)/e)^c \cup (G(X)/e)^c. \tag{3}$$

Conversely;

Let $1-a \in (F(x)/e)^c \cup (G(X)/e)^c$ it implies that $\max(\max((F(x)/e)^c), \max((G(x)/e)^c)) \geq 1-a \geq \min(\max((F(x)/e)^c), \max((G(x)/e)^c)$, $1-a \in (F(x)/e)^c$ and $1-a \in (G(x)/e)^c$.

It further implies that

$\max(1-\min(F(x)/e), 1-\min(G(x)/e)) \geq 1-a \geq \min(1-\min(F(x)/e), 1-\min(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

This inequality can be further written as $1-(\min(\min(F(x)/e), \min(G(x)/e))) \geq 1-a \geq 1-(\max(\min(F(x)/e), \min(G(x)/e)))$, $a \in F(x)/e$ and $a \in G(x)/e$.

Therefore

$\min(\min(F(x)/e), \min(G(x)/e)) \leq a \leq$

$\max(\min(F(x)/e), \min(G(x)/e))$, $a \in F(x)/e$ and $a \in G(x)/e$.

This shows that $a \in F(x)/e \cap G(x)/e$ which further implies that $1-a \in (F(x)/e \cap G(x)/e)^c$.

So

$$(F(x)/e)^c \cup (G(X)/e)^c \subseteq (F(x)/e \cap G(x)/e)^c \tag{4}$$

Equation. 3 and 4, imply that

$$(F(x)/e \cap G(x)/e)^c = (F(x)/e)^c \cup (G(X)/e)^c.$$

■

Example 3.6. Suppose that $X = \{a_1, a_2, a_3\}$ is a universal set and $E = \{e_1, e_2, e_3\}$ is a set of parameters.

Table 10
HFSS $F_1(X)$

	e_1	e_2	e_3
a_1	$\{0.5, 0.6, 0.7\}$	$\{0.2, 0.4\}$	$\{0.2, 0.35\}$
a_2	$\{0.75, 0.8\}$	$\{0.2, 0.25\}$	$\{0.4, 0.6, 0.7\}$
a_3	$\{0.9, 1\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Table 11
HFSS $F_2(X)$

	e_1	e_2	e_3
a_1	$\{0.4, 0.3\}$	$\{0.2\}$	$\{0.5, 0.6\}$
a_2	$\{0.8, 0.9, 1\}$	$\{0.2\}$	$\{0.8, 0.9\}$
a_3	$\{0.4, 0.5\}$	$\{0.3, 0.5\}$	$\{0.9, 1\}$

Table 12
HFSS $F_1(X) \cup F_2(X)$

	e_1	e_2	e_3
a_1	$\{0.4, 0.5, 0.6, 0.7\}$	$\{0.2, 0.4\}$	$\{0.2, 0.35\}$
a_2	$\{0.8, 0.9, 1\}$	$\{0.2, 0.25\}$	$\{0.4, 0.6, 0.7\}$
a_3	$\{0.5, 0.9, 1\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Table 13
HFSS $(F_1(X) \cup F_2(X))^c$

	e_1	e_2	e_3
a_1	$\{0.6, 0.5, 0.4, 0.3\}$	$\{0.2, 0.4\}$	$\{0.2, 0.35\}$
a_2	$\{0.2, 0.1, 0\}$	$\{0.2, 0.25\}$	$\{0.4, 0.6, 0.7\}$
a_3	$\{0.5, 0.1, 0\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Table 14
HFSS $(F_1(X))^c$

	e_1	e_2	e_3
a_1	$\{0.5, 0.4, 0.3\}$	$\{0.2, 0.4\}$	$\{0.2, 0.35\}$
a_2	$\{0.25, 0.2\}$	$\{0.2, 0.25\}$	$\{0.4, 0.6, 0.7\}$
a_3	$\{0.1, 0\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Let HFSSs $F_1(X)$ and $F_2(X)$ be given in Tables 10 and 11, respectively.

Union of HFSSs $F_1(X) \cup F_2(X)$ is given in Table 12.

Complement of HFSS $F_1(X) \cup F_2(X)$ is given in Table 13.

Complement of HFSSs $F_1(X)$ and $F_2(X)$ are shown in Tables 14 and 15, respectively.

Intersection of HFSSs $(F_1(X))^c$ and $(F_2(X))^c$ is given in Table 16. In this example, it is shown that these HFSSs $F_1(X)$ and $F_2(X)$ holds De Morgan's Laws.

Multi-criteria group decision making problem includes uncertain and imprecise data and information. First we give a distance notion for any two HFE's and then use this distance in construction of distance between any two HFSE's.

Table 15
HFSS $(F_2(X))^c$

	e_1	e_2	e_3
a_1	$\{0.6, 0.7\}$	$\{0.2\}$	$\{0.5, 0.6\}$
a_2	$\{0.2, 0.1, 0\}$	$\{0.2\}$	$\{0.8, 0.9\}$
a_3	$\{0.6, 0.5\}$	$\{0.3, 0.5\}$	$\{0.9, 1\}$

Table 16
HFSS $(F_1(X))^c \cap (F_2(X))^c$

	e_1	e_2	e_3
a_1	$\{0.6, 0.5, 0.4, 0.3\}$	$\{0.2, 0.4\}$	$\{0.2, 0.35\}$
a_2	$\{0.2, 0.1, 0\}$	$\{0.2, 0.25\}$	$\{0.4, 0.6, 0.7\}$
a_3	$\{0.5, 0.1, 0\}$	$\{0.1, 0.2, 0.3\}$	$\{0.5, 0.8\}$

Let $F(x)$ and $F(y)$ be the two HFSEs, then distance 'D' between $F(x)$ and $F(y)$ is defined as $D(F(x), F(y)) = \sum_{e_i \in E} d(F(x)/e_i, F(y)/e_i)$, where

$$d(F(x)/e_i, F(y)/e_i) = \max\left\{\begin{array}{c}\max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(y)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(x)/e_i}(|a-b|)\right\}\end{array}\right\}.$$

Theorem 3.7. *Distance 'd' satisfies the following four conditions.*

1. $d(F(x)/e_i, F(y)/e_i) = 0$ iff $F(x)/e_i = F(y)/e_i$;
2. $d(F(x)/e_i, F(y)/e_i) \geq 0$;
3. $d(F(x)/e_i, F(y)/e_i) = d(F(y)/e_i, F(x)/e_i)$;
4. $d(F(x)/e_i, F(z)/e_i) + d(F(z)/e_i, F(y)/e_i) \geq d(F(x)/e_i, F(y)/e_i)$.

Proof.

1. $d(F(x)/e_i, F(y)/e_i) = 0$

$$\Longleftrightarrow \max\left\{\begin{array}{c}\max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(y)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(x)/e_i}(|a-b|)\right\}\end{array}\right\} = 0$$

$$\Longleftrightarrow \max_{a\in F(x)/e_i}\left\{\min_{b\in F(y)/e_i}(|a-b|)\right\} = 0 \quad \text{and}$$

$$\max_{b\in F(y)/e_i}\left\{\min_{a\in F(x)/e_i}(|a-b|)\right\} = 0$$

$\Longleftrightarrow \min\limits_{b\in F(y)/e_i}(|a-b|) = 0$ for all $a \in F(x)/e_i$ and $\min\limits_{a\in F(x)/e_i}(|a-b|) = 0$ for all $b \in F(y)/e_i$

$\Longleftrightarrow$ For every a there exist b such that $a = b$ and for every b there exist a such that $b = a$.

$\Longleftrightarrow F(x)/e_i = F(y)/e_i$

2. Since $a, b \in [0, 1]$, then $|a-b| \geq 0$ and $|b-a| \geq 0$.

   As we know that the min and max of non-negative numbers is also non-negative, so

$$\max\left\{\begin{array}{l}\max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(y)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(x)/e_i}(|a-b|)\right\}\end{array}\right\} \geq 0.$$

   Thus, $d(F(x)/e_i, F(y)/e_i) \geq 0$.

3. Since $d(F(x)/e_i, F(y)/e_i)$

$$= \max\left\{\begin{array}{l}\max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(y)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(x)/e_i}(|a-b|)\right\}\end{array}\right\} \text{ and}$$

   we know that $\max(a, b) = \max(b, a)$.

   So, $d(F(x)/e_i, F(y)/e_i)$

$$= \max\left\{\begin{array}{l}\max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(x)/e_i}(|a-b|)\right\},\\ \max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(y)/e_i}(|a-b|)\right\}\end{array}\right\}$$

   Hence

   $d(F(x)/e_i, F(y)/e_i) = d(F(y)/e_i, F(x)/e_i)$.

4. Since $d(F(x)/e_i, F(y)/e_i)$

$$= \max\left\{\begin{array}{l}\max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(y)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(x)/e_i}(|a-b|)\right\}\end{array}\right\} \text{ and}$$

   we know that $\max(a, b) \leq \max(a, b, c)$ for all a, b, $c \in [0, 1]$.

   So,

   $d(F(x)/e_i, F(y)/e_i)$

$$\leq \max\left\{\begin{array}{l}\max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(z)/e_i}(|a-b|)\right\},\\ \max\limits_{a\in F(z)/e_i}\left\{\min\limits_{b\in F(x)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(z)/e_i}\left\{\min\limits_{a\in F(y)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(z)/e_i}(|a-b|)\right\}\end{array}\right\}$$

$$\leq \max\left\{\begin{array}{l}\max\limits_{a\in F(x)/e_i}\left\{\min\limits_{b\in F(z)/e_i}(|a-b|)\right\},\\ \max\limits_{a\in F(z)/e_i}\left\{\min\limits_{b\in F(x)/e_i}(|a-b|)\right\}\end{array}\right\} + \max\left\{\begin{array}{l}\max\limits_{b\in F(z)/e_i}\left\{\min\limits_{a\in F(y)/e_i}(|a-b|)\right\},\\ \max\limits_{b\in F(y)/e_i}\left\{\min\limits_{a\in F(z)/e_i}(|a-b|)\right\}\end{array}\right\}.$$

   Hence

   $d(F(x)/e_i, F(y)/e_i) \leq d(F(x)/e_i, F(z)/e_i) + d(F(z)/e_i, F(y)/e_i)$.

■

The following corollary can be proved very easily.

Corollary 3.8. *Distance 'D' satisfies the following four conditions.*

1. $D(F(x), F(y)) \geq 0$;
2. $D(F(x), F(y)) = 0$ if and only if $F(x) = F(y)$;
3. $D(F(x), F(y)) = D(F(y), F(x))$;
4. $D(F(x), F(y)) \leq D(F(x), F(z)) + D(F(z), F(x))$.

4. TOPSIS for HFSS

Next we give construction of TOPSIS using our notion of distance, which is then used for multi-criteria group decision making where the opinions of decision makers are expressed in HFSS. We suppose that in this group decision making problem, $DM = \{dm_1, dm_2, \ldots, dm_K\}$ is the set of the decision makers involved in the decision problem; $AL = \{al_1, al_2, \ldots, al_m\}$ is the set of the considered alternatives and $CR = \{cr_1, cr_2, \ldots, cr_n\}$ is the set of the criteria used for evaluating the alternatives.

Step 1. MCDM problem where performance of alternatives AL with respect to decision maker dm_l and criteria CR is denoted by HFSS $F_l(AL)$, in a group decision environment with K decision makers.

Step 2. We calculate the final decision in HFSS by aggregating the opinions of DMs.

$F(al_i)/cr_j = \{x \mid x \in F_l(al_i)/cr_j$ and $s_{p_{ij}} \leq x \leq s_{q_{ij}}$ for all $l\}$ where

$$s_{p_{ij}} = \min\left\{\begin{array}{l}\min\limits_{l=1}^{K}(\max F_l(al_i)/cr_j),\\ \max\limits_{l=1}^{K}(\min F_l(al_i)/cr_j)\end{array}\right\}$$

and

$$s_{q_{ij}} = \max\left\{\begin{array}{l}\min\limits_{l=1}^{K}(\max F_l(al_i)/cr_j),\\ \max\limits_{l=1}^{K}(\min F_l(al_i)/cr_j)\end{array}\right\}.$$

Performance of alternative al_i with respect to criterion cr_j is denoted as $F(al_i)/cr_j$.

Step 3. Let Ω_b be the collection of benefit criteria (i.e., the larger cr_j, the greater preference) and Ω_c be the collection of cost criteria (i.e., the smaller cr_j, the greater preference). The HFSS positive ideal alternative, denoted as $F(al^+) = \{F(al^+)/cr_1, F(al^+)/cr_2, ..., F(al^+)/cr_n\}$, and the HFSS negative ideal alternative, denoted as $F(al^-) = \{F(al^-)/cr_1, F(al^-)/cr_2, ..., F(al^-)/cr_n\}$, are defined as follows:

$$F(al^+)/cr_j = \begin{cases} 1 & ; \quad \text{if } j \in \Omega_b \\ 0 & ; \quad \text{if } j \in \Omega_c \end{cases}$$

and

$$F(al^-)/cr_j = \begin{cases} 0 & ; \quad \text{if } j \in \Omega_b \\ 1 & ; \quad \text{if } j \in \Omega_c \end{cases}$$

Step 4. Calculate the relative closeness coefficient (RC) of each alternative to the ideal solution as follows:

$$RC(al_i) = \frac{D(F(al_i), F(al^-))}{D(F(al_i), F(al^+)) + D(F(al_i), F(al^-))}, \tag{5}$$

$i = 1, 2, \ldots, m$.

Step 5. Rank all the alternatives al_i $(i = 1, 2, \ldots, m)$ according to the relative closeness coefficient $RC(al_i)$ of Equation (5) greater the value $RC(al_i)$, better the alternative al_i.

5. Illustrative example

In this section, we give an example by utilizing the method proposed in Section 4 to get the best alternative.

Step 1. Water and power development authority (WA-PDA) wants to purchase thermal power generation units. There are five types of units available with the following four criteria: cr_1 is the environmental pollution; cr_2 electricity generation; cr_3 is the warranty period, cr_4 is maintenance problems. Technical committee consists of three members to decide the unit type. The five possible units (alternatives) al_i $(i = 1, 2, 3, 4, 5)$ are to be evaluated using the HFSS by three committee members dm_K $(K = 1, 2, 3)$, as listed in Tables 17–19.

Step 2. The final aggregated decision Table 20 is constructed by utilizing Tables 17–19.

Table 17
Member 1 (dm_1) opinion by using HFSS

	cr_1	cr_2	cr_3	cr_4
al_1	{0.5, 0.6, 0.7}	{0.7, 0.8}	{0.2, 0.4}	{0.2, 0.35}
al_2	{0.2, 0.3}	{0.6, 0.7, 0.9}	{0.6, 0.7}	{0.45, 0.55}
al_3	{0.6, 0.8}	{0.4, 0.55}	{0.8, 0.95}	{0.2}
al_4	{0.75, 0.8}	{0.15, 0.23}	{0.2, 0.25}	{0.4, 0.6, 0.7}
al_5	{0.9, 1}	{0.12, 0.35}	{0.1, 0.2, 0.3}	{0.5, 0.8}

Table 18
Member 2 (dm_2) opinion by using HFSS

	cr_1	cr_2	cr_3	cr_4
al_1	{0.4, 0.3}	{0.6, 0.8}	{0.2}	{0.5, 0.6}
al_2	{0.1}	{0.2, 0.4}	{0.6, 0.7, 0.8}	{0.8, 1}
al_3	{0.2, 0.3}	{0, 0.3, 0.4}	{0.3, 0.5}	{0.3}
al_4	{0.8, 0.9, 1}	{0.6, 0.7}	{0.2}	{0.8, 0.9}
al_5	{0.4, 0.5}	{0.7}	{0.3, 0.5}	{0.9, 1}

Table 19
Member 3 (dm_3) opinion by using HFSS

	cr_1	cr_2	cr_3	cr_4
al_1	{0.2, 0.3}	{0.9, 1}	{0.5, 0.6}	{0.1, 0.4}
al_2	{0.5, 0.7}	{0.2, 0.3}	{0.7, 0.9}	{0.4, 0.5}
al_3	{0, 0.2, 0.3}	{0.7, 0.8}	{0, 0.2, 0.5}	{0.2}
al_4	{0.8, 1}	{0.4, 0.6}	{0.1, 0.5}	{0.7, 0.8, 1}
al_5	{0.4}	{0.1, 0.2, 0.3}	{0.3, 0.6}	{0.9, 1}

Table 20
Aggregated decision

	cr_1	cr_2	cr_3	cr_4
al_1	{0.3, 0.4, 0.5}	{0.8, 0.9}	{0.2, 0.4, 0.5}	{0.35, 0.4, 0.5}
al_2	{0.1, 0.2, 0.3, 0.5}	{0.3, 0.4, 0.6}	{0.7}	{0.5, 0.55, 0.8}
al_3	{0.3, 0.6}	{0.4, 0.55, 0.7}	{0.5, 0.8}	{0.2, 0.3}
al_4	{0.8}	{0.23, 0.4, 0.6}	{0.2}	{0.7, 0.8}
al_5	{0.4, 0.5, 0.9}	{0.3, 0.35, 0.7}	{0.3}	{0.8, 0.9}

Step 3. For cost criteria cr_1, cr_4 and benefit criteria cr_2, cr_3 HFSS positive ideal alternative $F(al^+)$ and HFSS negative ideal alternative $F(al^-)$ is as follows:

Ideal Solutions.

	cr_1	cr_2	cr_3	cr_4
$F(al^+)$	{0}	{1}	{1}	{0}
$F(al^-)$	{1}	{0}	{0}	{1}

Step 4. Relative closeness coefficient (RC) by equation (5) of each alternative to the ideal solutions:

$RC(al_1) = 2.75/(2.75 + 2) = 0.5789$;
$RC(al_2) = 2.7/(2.7 + 2.3) = 0.54$;
$RC(al_3) = 3/(3 + 2) = 0.6$;
$RC(al_4) = 1.3/(1.3 + 3.17) = 0.2908$;
$RC(al_5) = 1.8/(1.8 + 3.2) = 0.36$.

Step 5. Rank all the alternatives al_i $(i = 1, 2, \ldots, 5)$ according to the relative closeness coefficient $RC(al_i)$:

$al_3 \succ al_1 \succ al_2 \succ al_4 \succ al_5$.

Thus the most desirable alternative is al_3.

6. Conclusion

In this paper we describe the HFSS in a new way and an extended fuzzy TOPSIS method is proposed for solving multi-criteria decision making problem with the opinion of some experts in HFSS. TOPSIS is mainly based on distance measure to calculate the relative closeness coefficient of alternatives. Our proposed distance measure is also metric. It will be very helpful to discuss various topological structures for HFSS. The previous decision making technique for HFSS is based on level sets [18]. Here TOPSIS is given to rank the alternatives from best to worst or vice versa. An example is given for the ranking of alternatives to show the significance of our proposed decision making procedure. This newly proposed method is based on fuzzy TOPSIS for HFSS. In future we plan to study TOPSIS for HFSS with the use of Choquet integral and the interaction phenomena about the criteria are also under consideration.

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