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RESEARCH ARTICLE

Self-gravitating spherically symmetric fluid models in Brans–Dicke gravity

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Abstract This paper is devoted to study self-gravitating spherically symmetric fluid models in Brans–Dicke gravity. We formulate a set of equations which govern the dynamics of evolving gravitating fluids through Weyl tensor, shear tensor, expansion scalar, anisotropy, energy inhomogeneity, dissipation as well as scalar field. We also discuss some particular cases according to different dynamical conditions. It is concluded that fluid models for regular distribution of scalar field are consistent with general relativity and models due to irregular distribution of scalar field deviate from theory of general relativity.

Keywords Brans–Dicke theory · Relativistic dissipative fluid

1 Introduction

The phenomenon of gravitational collapse is a process through which stars, planets and cluster of galaxies born in the universe. This fact attracted many researchers to explore the dynamics of gravitational collapse, in particular, the modeling of evolving self-gravitating fluids in general relativity (GR). Dynamical variables (physical and kinematical variables) are considered to be fundamental tool for the evolution of any

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self-gravitating fluid. These variables include density inhomogeneity, shear tensor, local anisotropy of pressure, dissipation and Weyl tensor etc out of which density inhomogeneity plays a vital role in the collapse of self-gravitating fluid [1–6].

Penrose [7] described the effects of Weyl tensor on energy density inhomogeneity during the evolution and collapse of self-gravitating fluid. Herrera [8] studied the role of Weyl tensor in the stability of spherically symmetric fluid. There has been a large body of literature [9–14] which indicates the importance of anisotropic pressure in different self-gravitating fluid models. The presence of shear tensor has also a crucial role in the evolution of self-gravitating fluid [15–20]. Since gravitational collapse is a highly dissipative process, so the relevance of dissipation in self-gravitating objects has also attracted many people [21–25]. Herrera et al. [26,27] studied spherically symmetric self-gravitating dissipative fluid by involving all dynamical variables. Sharif and Bhatti [28,29] constructed models of dissipative self-gravitating fluid in plane and cylindrically symmetric spacetimes. Herrera [30] discussed the role of dissipation, Weyl tensor and vorticity in the axially symmetric fluid model.

The mystery of cosmic accelerated expansion induces the concept of correct theory of gravity. In this context, various modified theories of gravity are introduced by modifying the Einstein–Hilbert action. In weak-field regimes, all theories of gravity are consistent with GR but in strong-field regimes they may deviate widely from GR. It is believed that modified theories in strong-field regimes may lead to a suitable theory of gravity. The phenomenon of gravitational collapse is a prominent example of strong-field regime, so its analysis in modified theories has attracted many researchers [31–44]. The modeling of self-gravitating fluid distribution in generalized theories of gravity may modify the dynamics of collapse which in turn reveals the modification hidden in the structure formation of the universe [45,46].

Brans–Dicke (BD) theory, being a scalar-tensor theory, provides considerable deviation from GR in strong field regimes [39–44,47–49]. This theory is a generalized form of GR, where gravitational field is mediated by a relation between scalar field ϕ and field due to geometric part (R). Also, it contains tuneable constant coupling parameter ω_{BD} which can be set to fit the observations [47–49]. It has been proved (by radar timing experiment) that BD gravity is consistent with all solar system observations and experiments for $|\omega_{BD}| > 40{,}000$ [50,51]. This theory also explains structure formation in the early universe (radiation-matter transition) by incorporating observational data like cosmic microwave background data from WMAP, VSA, CBI and two degree galaxy red shift survey [52] which provides a wide range of deviation from GR [53–55]. It also provides convenient evidences of many cosmic issues such as cosmic acceleration, inflation, late behavior of the universe and coincidence problem [56–59].

In this paper, we provide a detail description of spherically symmetric model of self-gravitating fluid in BD gravity. The paper is arranged as follows. The next section provides BD field equations and kinematic variables. Section 3 explores the set of governing equations of the evolving distribution. In Sect. 4, we discuss some particular fluid models. Finally, Sect. 5 summarizes the results.

2 Brans–Dicke equations and dynamical variables

In this section, we write BD equations for a spherically symmetric anisotropic dissipative fluid and then discuss the dynamical variables [26] of the respective fluid. The action of BD gravity with self-interacting potential $V(\phi)$ $(8\pi G_0 = c = 1)$ is [47–49]

$$S = \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega_{BD}}{\phi}\nabla^{\alpha}\phi\nabla_{\alpha}\phi - V(\phi) + L_m\right], \tag{1}$$

where G_0 is the present day value of gravitational constant and L_m expresses the matter contribution, respectively. Variation of this action provides the following BD equations

$$G_{\mu\nu} = T^{(eff)}_{\mu\nu} = \frac{1}{\phi}\left(T^m_{\mu\nu} + T^{\phi}_{\mu\nu}\right), \tag{2}$$

$$\Box\phi = \frac{T^m}{3+2\omega_{BD}} + \frac{1}{3+2\omega_{BD}}\left[\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right], \tag{3}$$

where $T^m_{\mu\nu}$ $(\mu,\ \nu = 0, 1, 2, 3)$ represents the energy-momentum tensor of matter distribution, $T^m = g^{\mu\nu}T^m_{\mu\nu}$, $\Box$ shows the d'Alembertian operator and

$$T^{\phi}_{\mu\nu} = \left[\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi\right] + \frac{\omega_{BD}}{\phi}\left[\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right] - \frac{V(\phi)}{2}g_{\mu\nu}, \tag{4}$$

is the energy part due to scalar field. Equation (2) represents the field equations in BD gravity and (3) is a wave equation which describes the evolution of scalar field.

We take spherically symmetric dissipative fluid distribution bounded by a spherical surface Σ as

$$ds^2 = -A^2(t,r)dt^2 + B^2(t,r)dr^2 + C^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where t and r are comoving coordinates. The matter distribution is

$$T^m_{\mu\nu} = \rho u_{\mu}u_{\nu} + p_{\perp}h_{\mu\nu} + (p_r - p_{\perp})\chi_{\mu}\chi_{\nu} + q(\chi_{\mu}u_{\nu} + u_{\mu}\chi_{\nu}) + \epsilon l_{\mu}l_{\nu}. \tag{6}$$

Here $\rho,\ p_{\perp},\ p_r,\ q$ and ϵ represent the energy density, tangential pressure, radial pressure, heat flux and radiation density, respectively. The four velocity, u^{μ}, a unit four-vector χ^{μ} (along radial direction), the null four-vector l_{μ} and projection tensor $h_{\mu\nu}$ are calculated as $u^{\mu} = A^{-1}\delta^{\mu}_0$, $\chi^{\mu} = B^{-1}\delta^{\mu}_1$, $l^{\mu} = A^{-1}\delta^{\mu}_0 + B^{-1}\delta^{\mu}_1$ and $h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$. These quantities satisfy

$$u^{\mu}u_{\mu} = -1, \quad \chi^{\mu}\chi_{\mu} = 1, \quad \chi^{\mu}u_{\mu} = 0, \quad l^{\mu}u_{\mu} = -1, \quad l^{\mu}l_{\mu} = 0, \quad h_{\mu\nu}u^{\nu} = 0.$$

Equation (6) can be rewritten as

$$T^m_{\mu\nu} = \tilde{\rho}u_{\mu}u_{\nu} + p_{\perp}h_{\mu\nu} + \Pi\chi_{\mu}\chi_{\nu} + \tilde{q}_{\mu}u_{\nu} + \tilde{q}_{\nu}u_{\mu}, \tag{7}$$

where

$$\tilde{\rho}=\rho+\epsilon,\quad \Pi=\tilde{p}_r-p_\perp,\quad \tilde{p}_r=p_r+\epsilon,\quad \tilde{q}_\mu=\tilde{q}\chi_\mu=(q+\epsilon)\chi_\mu. \tag{8}$$

For the metric (5), the BD field equations become

$$\begin{aligned}
&\left(\frac{2\dot{B}}{B}+\frac{\dot{C}}{C}\right)\frac{\dot{C}}{C}-\left(\frac{A}{B}\right)^2\left[\frac{2C''}{C}+\left(\frac{C'}{C}\right)^2-\frac{2B'C'}{BC}-\left(\frac{B}{C}\right)^2\right]\\
&\quad=\frac{1}{\phi}(T^m_{00}+T^\phi_{00})=\frac{1}{\phi}\left(\tilde{\rho}A^2+\frac{\omega_{BD}}{2\phi}\left(\dot{\phi}^2+\frac{A^2\phi'^2}{B^2}\right)\right)-\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right.\\
&\qquad\left.+\frac{2\dot{C}}{C}\right)+\frac{\phi'}{\phi B^2C}\left(\frac{A^2B'C}{B}+2AA'C+2A^2C'\right)+\frac{A^2\phi''}{B^2\phi}-\frac{A^2V(\phi)}{2\phi},
\end{aligned}\tag{9}$$

$$\begin{aligned}
2\left(-\frac{\dot{C}'}{C}+\frac{\dot{C}A'}{CA}+\frac{\dot{B}C'}{BC}\right)&=\frac{1}{\phi}(T^m_{01}+T^\phi_{01})=\frac{\tilde{q}AB}{\phi}+\frac{\omega_{BD}}{\phi^2}(\dot{\phi}\phi')\\
&\quad+\frac{1}{\phi}\left(\dot{\phi}'-\frac{\dot{A}\dot{\phi}}{A}-\frac{\dot{B}\phi'}{B}\right),
\end{aligned}\tag{10}$$

$$\begin{aligned}
&-\left(\frac{B}{A}\right)^2\left[\frac{2\ddot{C}}{C}-\left(\frac{2\dot{A}}{A}-\frac{\dot{C}}{C}\right)\frac{\dot{C}}{C}\right]+\left(\frac{2A'}{A}+\frac{C'}{C}\right)\frac{C'}{C}-\left(\frac{B}{C}\right)^2\\
&\quad=\frac{1}{\phi}(T^m_{11}+T^\phi_{11})=\frac{1}{\phi}\left(\tilde{p}_rB^2+\frac{\omega_{BD}}{2\phi}\left(\phi'^2+\frac{B^2\dot{\phi}^2}{A^2}\right)\right)+\frac{\dot{\phi}}{\phi A^2C}\left(\frac{A\dot{C}B^2}{A}\right.\\
&\qquad\left.+2B\dot{B}C\frac{2\dot{C}B^2}{C}\right)+\frac{\phi'}{\phi}\left(\frac{B'}{B}+\frac{A'}{A}+\frac{2C'}{C}+\frac{\dot{B}}{B}\right)+\frac{B^2V(\phi)}{2\phi},
\end{aligned}\tag{11}$$

$$\begin{aligned}
&-\left(\frac{C}{A}\right)^2\left[\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\frac{\dot{B}\dot{C}}{BC}\right]+\left(\frac{C}{B}\right)^2\left[\frac{A''}{A}+\frac{C''}{C}\right.\\
&\quad\left.-\frac{A'B'}{AB}+\left(\frac{A'}{A}-\frac{B'}{B}\right)\frac{C'}{C}\right]=\frac{1}{\phi}(T^m_{22}+T^\phi_{22})=\frac{1}{\phi}(T^m_{33}+T^\phi_{33})\\
&=\frac{1}{\phi}\left(p_\perp C^2+\frac{\omega_{BD}}{2\phi}\left(\frac{\dot{C}^2\dot{\phi}^2}{A^2}-\frac{C^2\phi'^2}{B^2}\right)\right)+\frac{\dot{\phi}}{A^2B\phi}\left(\dot{A}ABC^2+\dot{B}C^2+C\dot{C}B\right)\\
&\quad-\frac{\phi'}{AB^2\phi}\left(\frac{B'AC^2}{B}+A'C^2+3CC'A\right)+\frac{\ddot{\phi}C^2}{A^2\phi}+\frac{C^2V(\phi)}{2\phi},
\end{aligned}\tag{12}$$

and the respective wave equation takes the form

$$\begin{aligned}
\Box\phi&=\dot{\phi}\left(-\frac{\dot{A}}{A}+\frac{\dot{B}}{A^2B}+\frac{2\dot{C}}{A^2B}\right)+\frac{\ddot{\phi}}{A^2}+\phi'\left(-\frac{A'}{AB^2}+\frac{B'}{B^3}-\frac{2C'}{CB^2}\right)-\frac{\phi''}{B^2}\\
&=\frac{1}{2\omega_{BD}+3}\left[(-\tilde{\rho}+\tilde{p}_r+2p_\perp)+\left(\phi\frac{dV(\phi)}{d\phi}-2V(\phi)\right)\right].
\end{aligned}\tag{13}$$

Here dot and prime indicate derivatives with respect to t and r, respectively.

The four acceleration a_μ, expansion scalar Θ and shear tensor $\sigma_{\mu\nu}$ are defined by

$$a_\mu = u_{\mu;\nu}u^\nu, \quad \Theta = u^\mu_{;\mu}, \quad \sigma_{\mu\nu} = u_{(\mu;\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{3}\Theta h_{\mu\nu}, \tag{14}$$

which provide

$$a_{(1)} = \frac{A'}{A}, \quad a^2 = a^\mu a_\mu = \left(\frac{A'}{AB}\right)^2, \quad a_\mu = a\chi_\mu, \quad \Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C}\right). \tag{15}$$

The shear tensor can also be expressed in terms of four-vector and projection tensor as

$$\sigma_{\mu\nu} = \sigma(\chi_\mu\chi_\nu - \frac{1}{3}h_{\mu\nu}), \tag{16}$$

where

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \quad \sigma^{\mu\nu}\sigma_{\mu\nu} = \frac{2}{3}\sigma^2. \tag{17}$$

The non-zero components of shear tensor are

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \quad \sigma_{22} = \frac{\sigma_{33}}{\sin^2\theta} = -\frac{C^2\sigma}{3}.$$

The Weyl tensor describes tidal forces experienced by a body moving along geodesics in the region free from matter and is defined as follows

$$C^\rho_{\mu\nu\sigma} = R^\rho_{\mu\nu\sigma} - \frac{1}{2}R^\rho_\nu g_{\mu\sigma} + \frac{1}{2}R_{\mu\nu}\delta^\rho_\sigma - \frac{1}{2}R_{\mu\sigma}\delta^\rho_\nu + \frac{1}{2}R^\rho_\sigma g_{\mu\nu} + \frac{1}{6}\mathcal{R}(\delta^\rho_\nu g_{\mu\sigma} - g_{\mu\nu}\delta^\rho_\sigma), \tag{18}$$

where $R^\rho_{\mu\nu\sigma}$, R^ρ_ν and $\mathcal{R}$ represent Riemann tensor, Ricci tensor and Ricci scalar, respectively. It is divided into electric $E_{\mu\nu}$ and magnetic parts $H_{\mu\nu}$. For spherically symmetric distribution, the magnetic part vanishes while the electric is given by

$$E_{\mu\nu} = C_{\mu\rho\nu\sigma}u^\rho u^\sigma. \tag{19}$$

Here

$$C_{\alpha\beta\kappa\eta} = (g_{\alpha\beta\mu\nu}g_{\kappa\eta\gamma\delta} - \varepsilon_{\alpha\beta\mu\nu}\varepsilon_{\kappa\eta\gamma\delta})u^\mu u^\gamma E^{\nu\delta}, \quad g_{\alpha\beta\mu\nu} = g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}, \tag{20}$$

and $\varepsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor. The Weyl tensor can also be written in terms χ_μ and $h_{\mu\nu}$ as

$$E_{\mu\nu} = \varepsilon\left(\chi_\mu\chi_\nu - \frac{1}{3}h_{\mu\nu}\right), \tag{21}$$

where

$$\varepsilon=\frac{1}{2}\left[\frac{\ddot{C}}{C}-\frac{\ddot{B}}{B}-\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right)\frac{\dot{C}}{C}\right]+\frac{1}{2B^2}\left[-\frac{C''}{C}+\left(\frac{B'}{B}+\frac{C'}{C}\right)\frac{C'}{C}\right]-\frac{1}{2C^2},$$

and its non-vanishing components are

$$E_{11}=\frac{2}{3}B^2\varepsilon,\quad E_{22}=-\frac{1}{3}C^2\varepsilon,\quad E_{33}=E_{22}\sin^2\theta.$$

The Misner-Sharp mass function is defined by

$$m=\frac{C^3}{2}R^{23}_{23}=\frac{C}{2}\left[\left(\frac{\dot{C}}{A^2}\right)^2-\left(\frac{C'}{B}\right)^2+1\right], \tag{22}$$

which evaluates total energy of a spherical distribution within radius C. The velocity of the collapsing fluid is evaluated with the help of proper time derivative as

$$U=D_TC=\frac{1}{A}\frac{\partial C}{\partial t}=\frac{\dot{C}}{A}, \tag{23}$$

which along with (22) yields

$$E=\frac{C'}{B}=\sqrt{1+U^2-\frac{2m}{C}}. \tag{24}$$

The radial derivative for radius C (inside the surface Σ) is given by

$$D_C=\frac{1}{C'}\frac{\partial}{\partial r},$$

and the rates of change of mass function with respect to proper time T and C are obtained from Eq. (24) and the field Eqs. (9)–(11) as

$$D_Tm=\frac{-1}{2\phi}\left[\left(\tilde{p}_r+\frac{T^{\phi}_{11}}{B^2}\right)U+E\left(\tilde{q}-\frac{T^{\phi}_{01}}{AB}\right)\right]C^2, \tag{25}$$

$$D_Cm=\frac{1}{2\phi}\left[\tilde{\rho}+\frac{T^{\phi}_{00}}{A^2}+\frac{U}{E}\left(\tilde{q}-\frac{T^{\phi}_{01}}{AB}\right)\right]C^2. \tag{26}$$

Equation (25) represents effects of pressure, dissipation, velocity and scalar field on the variation of energy within the evolving spherical distribution. Equation (26) shows the effects of energy density, dissipation, velocity and scalar field on the mass function within the surface of radius C. From Eq. (26), we obtain

$$m'=\frac{1}{2\phi}\left[\tilde{\rho}+\frac{T^{\phi}_{00}}{A^2}+\frac{U}{E}\left(\tilde{q}-\frac{T^{\phi}_{01}}{AB}\right)\right]C^2C',$$

which on integration gives

$$m=\int_0^r\frac{1}{2\phi}\left[\tilde{\rho}+\frac{T_{00}^{\phi}}{A^2}+\frac{U}{E}\left(\tilde{q}-\frac{T_{01}^{\phi}}{AB}\right)\right]C^2C'dr.$$

3 Evolution equations

Here, we formulate governing equations [26,27] of self-gravitating anisotropic and dissipative fluid.

3.1 Evolution equation for the scalar field

Equation (13) represents a wave equation which describes evolution of scalar field by taking into account energy density, radial and tangential pressures.

3.2 Dynamical equations

Dynamical equations of any evolving star describe the conservation of total energy of the star and are obtained from the contracted Bianchi identities as

$$\left(\frac{T_m^{\mu\nu}}{\phi}+\frac{T_{\phi}^{\mu\nu}}{\phi}\right)_{;\nu}u_{\mu}=0,\quad\left(\frac{T_m^{\mu\nu}}{\phi}+\frac{T_{\phi}^{\mu\nu}}{\phi}\right)_{;\nu}\chi_{\mu}=0,\tag{27}$$

yielding

$$\left[\tilde{\rho}_{,\mu}u^{\mu}+(\tilde{\rho}+\tilde{p}_r)\Theta-\frac{2}{3}\Pi(\Theta+\sigma)+\tilde{q}_{,\mu}\chi^{\mu}+\tilde{q}\frac{2C'}{BC}+2\tilde{q}a\right]+H_1^{\phi}=0,\tag{28}$$

$$\left[\tilde{p}_{r,\mu}\chi^{\mu}+(\tilde{\rho}+\tilde{p}_r)a+\Pi\frac{2C'}{BC}+\tilde{q}_{,\mu}\chi^{\mu}+\frac{4}{3}\tilde{q}\Theta+\frac{2}{3}\tilde{q}\sigma\right]+H_2^{\phi}=0,\tag{29}$$

where H_1^{ϕ} and H_2^{ϕ} are energy terms due to scalar field given in Appendix **A**. The above equations show the effect of scalar field in the evolution of energy density and radial pressure.

3.3 Raychaudhuri equation

The Ricci identities for four velocity is given by

$$u_{\mu;\nu;\alpha}-u_{\mu;\alpha;\nu}=R_{\mu\nu\alpha}^{\beta}u_{\beta},$$

where

$$u_{\mu;\nu}=-a_{\mu}u_{\nu}+\sigma_{\mu\nu}+\frac{1}{3}\Theta h_{\mu\nu}, \tag{30}$$

$$\frac{1}{2}R^{\beta}_{\mu\nu\alpha}u_{\beta}=a_{\mu;[\alpha}u_{\nu]}+a_{\mu}u_{[\nu;\alpha]}+\frac{1}{3}\Theta_{,[\alpha}h_{\nu]\mu}+\frac{1}{3}\Theta h_{\mu[\nu;\alpha]}. \tag{31}$$

Contracting Eq. (31) with $u^{\nu}u^{\mu}$ and then indices μ and α, we obtain

$$\Theta_{,\mu}u^{\mu}+\frac{\Theta^{2}}{3}-\sigma^{\mu\nu}\sigma_{\mu\nu}-a^{\mu}_{;\mu}=-u_{\alpha}u^{\mu}R^{\alpha}_{\mu}, \tag{32}$$

which along with Eq. (2) gives

$$\Theta_{,\mu}u^{\mu}+\frac{\Theta^{2}}{3}-\frac{2}{3}\sigma^{2}-a^{\mu}_{;\mu}=\frac{1}{\phi}\Bigg[-\frac{1}{2}(\tilde{\rho}+3\tilde{p}_{r})+\Pi-\phi^{;\mu}_{;\nu}u_{\mu}u^{\nu}-\phi^{,\mu}\phi_{,\nu}u_{\mu}u^{\nu}\times\frac{\omega_{BD}}{\phi}+\frac{1}{2}(\Box\phi+V(\phi))\Bigg]. \tag{33}$$

This equation represents Raychaudari equation in BD gravity which describes the variation of distance between two particles in the respective gravity.

3.4 Constraint equation

In order to have a direct relation among shear tensor, expansion scalar, dissipation and energy terms of scalar field, we obtain a constraint equation by contracting indices μ and α in Eq. (31) and then contracting the resultant equation with $h^{\mu\nu}\chi_{\mu}$ as follows

$$\left(\frac{\sigma}{2}+\Theta\right)_{,\mu}\chi^{\mu}=-\frac{3\sigma C'}{2BC}-\frac{3\tilde{q}}{2\phi B}-\frac{3A}{2B\phi}\left(\Theta_{,t;r}+\frac{\omega_{BD}\phi^{,t}\phi_{,r}}{\phi}\right). \tag{34}$$

3.5 Propagation equation of shear

For propagation equation of shear, we first contract Eq. (31) with $u^{\nu}h^{\mu}_{\gamma}h^{\alpha}_{\delta}$ so that

$$u_{\beta}u^{\nu}R^{\beta}_{\mu\nu\alpha}h^{\mu}_{\gamma}h^{\alpha}_{\delta}=h^{\mu}_{\gamma}h^{\alpha}_{\delta}\left(a_{\mu;\alpha}-\sigma_{\mu\alpha;\nu}u^{\nu}\right)-a_{\gamma}a_{\delta}-u^{\nu}_{;\alpha}h^{\alpha}_{\delta}\left(\sigma_{\gamma\nu}+\frac{\Theta}{3}h_{\delta\nu}\right)-\frac{\Theta_{;\mu}u^{\mu}h_{\gamma\delta}}{3}. \tag{35}$$

Using Eq. (18), we rewrite the Riemann tensor in terms of Weyl tensor as

$$R^{\rho}_{\mu\nu\sigma}=C^{\rho}_{\mu\nu\sigma}+\frac{1}{2}R^{\rho}_{\nu}g_{\mu\sigma}-\frac{1}{2}R_{\mu\nu}\delta^{\rho}_{\sigma}+\frac{1}{2}R_{\mu\sigma}\delta^{\rho}_{\nu}-\frac{1}{2}R^{\rho}_{\sigma}g_{\mu\nu}-\frac{1}{6}\mathcal{R}(\delta^{\rho}_{\nu}g_{\mu\sigma}-g_{\mu\nu}\delta^{\rho}_{\sigma}), \tag{36}$$

which on contracting with $u_\beta u^\nu h^\mu_\gamma h^\alpha_\delta \chi^\gamma \chi^\delta$ yields

$$R^{\beta}_{\mu\nu\alpha} u_\beta u^\nu h^\mu_\gamma h^\alpha_\delta \chi^\gamma \chi\delta = \frac{1}{\phi}\left[\varepsilon + \Pi - \frac{1}{2}(\tilde{\rho} + 3\tilde{p}_r) - \frac{1}{2}\left[\frac{\phi_{,t;t}}{A^2} + \frac{\phi_{,r;r}}{B^2}\right.\right.$$
$$\left.\left. + \frac{\omega_{BD}}{\phi}\left(\frac{\dot{\phi}^2}{A^2} + \frac{\phi'^2}{B^2}\right) + \frac{V(\phi)}{6} + \frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{6\phi}\right]\right]. \tag{37}$$

Contracting Eq. (35) with $\chi^\gamma \chi^\delta$ and then from (33) and (37), we get

$$a_{,\mu}\chi^\mu - \sigma_{,\mu}u^\mu + a^2 - \frac{\sigma^2}{3} - \frac{2\Theta\sigma}{3} - \frac{aC'}{BC} = \frac{1}{\phi}\left[\varepsilon - \frac{\Pi}{2} + \frac{\Box\phi}{4} - \frac{V(\phi)}{4}\right.$$
$$\left. - \frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{4\phi} + \frac{\phi_{,t;t}}{4A^2} + \frac{\phi_{,r;r}}{4B^2} - \frac{\omega_{BD}}{\phi}\left(\frac{3\dot{\phi}^2}{2A^2} + \frac{\phi'^2}{B^2}\right)\right]. \tag{38}$$

This is the required equation which shows the effect of scalar field along with other dynamical variables on the evolution of shear for self-gravitating fluid in BD gravity.

3.6 Evolution equations for the Weyl tensor

Bainchi identities, $R_{\alpha\beta\kappa\delta;\eta} + R_{\alpha\beta\eta\kappa;\delta} + R_{\alpha\beta\delta\eta;\kappa} = 0$, provide a relation between the Weyl tensor and Ricci tensor given by

$$C^{\eta}_{\alpha\beta\kappa;\eta} = R_{\kappa[\alpha;\beta]} - \frac{1}{6} g_{\kappa[\alpha} R_{,\beta]}. \tag{39}$$

Using Eq. (2), this can be converted into a relation between the Weyl tensor and the effective energy-momentum tensor $T^{(eff)}_{\mu\nu}$ as

$$C^{\eta}_{\alpha\beta\kappa;\eta} = T^{(eff)}_{\kappa[\alpha;\beta]} - \frac{1}{6} g_{\kappa[\alpha} T^{(eff)}_{,\beta]}. \tag{40}$$

Equation (20) yields

$$u^\beta C^{\eta}_{\alpha\beta\kappa;\eta} + u^\beta_{;\eta} C^{\eta}_{\alpha\beta\kappa} = \Theta E_{\alpha\kappa} + u^\mu E_{\alpha\kappa;\mu} - u_{\kappa;\eta} E^\eta_\alpha - u_\kappa E^\eta_{\alpha;\eta}, \tag{41}$$

which when contracted with $h^\alpha_\mu h^\kappa_\nu u^\beta \chi^\mu \chi^\nu$ leads to

$$h^\alpha_\mu h^\kappa_\nu u^\beta \chi^\mu \chi^\nu C^{\eta}_{\alpha\beta\kappa;\eta} = \frac{4\Theta}{3} E_{\mu\nu}\chi^\mu\chi^\nu - u_{\nu;\eta} E^\eta_\mu \chi^\mu \chi^\nu + u^\beta E_{\alpha\kappa;\beta} h^\alpha_\mu h^\kappa_\nu \chi^\mu \chi^\nu$$
$$+ h_{\mu\nu}\sigma^{\kappa\beta} E_{\kappa\beta}\chi^\mu\chi^\nu - \sigma_{\kappa\mu} E^\kappa_\nu \chi^\mu \chi^\nu - \sigma^{\kappa\nu} E^\kappa_\mu \chi^\mu \chi^\nu. \tag{42}$$

The effective energy-momentum tensor gives

$$\begin{aligned} h^{\alpha}_{\mu}h^{\kappa}_{\nu}u^{\beta}T^{(eff)}_{\kappa\alpha;\beta} &= \frac{1}{\phi}\Bigg[(p_{\perp})_{;\beta}u^{\beta}h_{\mu\nu} + (\Pi\chi_{\kappa}\chi_{\alpha})_{;\beta}u^{\beta}h^{\kappa}_{\nu}h^{\alpha}_{\mu} + \tilde{q}_{\mu}a_{\nu} \\ &\quad + \tilde{q}_{\nu}a_{\mu} + \phi_{,\kappa;\alpha;\beta}u^{\beta}h^{\kappa}_{\nu}h^{\alpha}_{\mu} + \omega_{BD}\left(\frac{\phi_{,\kappa}\phi_{,\alpha}}{\phi}\right)_{;\beta}u^{\beta}h^{\kappa}_{\nu}h^{\alpha}_{\mu} \\ &\quad - h_{\mu\nu}\left[\Box\phi \quad + \frac{\omega_{BD}\phi^{'\mu}\phi_{,\mu}}{2} + \frac{V}{2}\right]_{;\beta}u^{\beta}\Bigg], \end{aligned} \tag{43}$$

$$\begin{aligned} h^{\alpha}_{\mu}h^{\kappa}_{\nu}u^{\beta}T^{(eff)}_{\kappa\beta;\alpha} &= \frac{1}{\phi}\Bigg[(-\tilde{\rho} + p_{\perp})\left(\sigma_{\mu\nu} + \frac{\Theta h_{\mu\nu}}{3}\right) - \tilde{q}_{;\alpha}h^{\alpha}_{\mu}\chi_{\nu} \\ &\quad + \Pi_{;\beta}u^{\beta}\chi_{\kappa}\chi_{\beta;\alpha}u^{\beta}h^{\alpha}_{\mu}h^{\kappa}_{\nu} + \phi_{,\kappa;\beta;\alpha}u^{\beta}h^{\alpha}_{\mu}h^{\kappa}_{\nu} \\ &\quad + \left(\frac{\omega_{BD}\phi_{,\kappa}\phi_{,\beta}}{\phi}\right)_{;\alpha}u^{\beta}h^{\alpha}_{\mu}h^{\kappa}_{\nu}\Bigg], \end{aligned} \tag{44}$$

$$\begin{aligned} h^{\alpha}_{\mu}h^{\kappa}_{\nu}u^{\beta}g_{[\alpha}T^{(eff)}_{;\beta]} &= h_{\mu\nu}\left[-\tilde{\rho} + 3p_{\perp} + \Pi\right]_{;\beta}u^{\beta} \\ &\quad - h_{\mu\nu}\left[3\Box\phi + 2V(\phi) + \frac{\omega_{BD}\phi_{,\mu}\phi^{,\mu}}{\phi}\right]_{;\beta}u^{\beta}. \end{aligned} \tag{45}$$

From Eqs. (40), (42)–(45), we obtain

$$\begin{aligned} &\Bigg[\left(\varepsilon + \frac{\tilde{\rho}}{2} - \frac{\Pi}{2} - \frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{4\phi} + \frac{V(\phi)}{4}\right)_{,\mu}u^{\mu} + \left(\varepsilon + \frac{\tilde{\rho}}{2} - \frac{\Pi}{2} + \tilde{p}_{r}\right)(\Theta + \sigma) \\ &+ \frac{3\tilde{q}C'}{2BC} + \frac{3\omega_{BD}}{2AB}\left(\frac{\phi'\dot{\phi}}{\phi}\right)_{;\mu}\chi^{\mu} + \frac{3}{2B^{2}}\left(\frac{\phi^{'2}}{\phi}\right)_{;\mu}u^{\mu}\Bigg] = 0. \end{aligned} \tag{46}$$

Similarly, contraction of Eq. (40) with $u^{\kappa}u^{\beta}h^{\alpha}_{\mu}\chi^{\mu}$ yields

$$\begin{aligned} &\left(\varepsilon + \frac{\tilde{\rho}}{2} - \frac{\Pi}{2} + \frac{\omega_{BD}\phi^{\mu}\phi_{\mu}}{2\phi} + \frac{\Box\phi}{4} + \frac{V(\phi)}{4}\right)_{,\mu}\chi^{\mu} + \frac{3\omega_{BD}}{2A^{2}}\left(\frac{\dot{\phi}^{2}}{\phi}\right)_{;\alpha}\chi^{\alpha} \\ &- \frac{3\omega_{BD}}{2AB}\left(\frac{\dot{\phi}\phi'}{\phi}\right)_{;\alpha}u^{\alpha} - \frac{3C'}{BC}\left(\frac{\Pi}{2} - \varepsilon\right) - \tilde{q}(\sigma + \Theta) = 0. \end{aligned} \tag{47}$$

Equations (46) and (47) describe relations among the Weyl tensor, density inhomogeneity, dissipation, anisotropy due to scalar field gravitational force.

3.7 Relation between Weyl tensor, Mass function and scalar field

From Eqs. (2), (22) and (36), we have

$$\frac{3m}{C^3} = \frac{1}{\phi}\left[\frac{\rho}{2} - \frac{\Pi}{2} - \varepsilon + \frac{1}{2}\left[3\Box\phi + \frac{\omega_{BD}\phi_{,c}\phi^{,c}}{\phi} + 2V(\phi)\right]\right]. \tag{48}$$

Equations (13), (28), (29), (33), (34), (38) and (46)–(48) represent the governing equations of dissipative anisotropic self-gravitating fluid model in BD gravity.

4 Particular cases

Here, we analyze the governing equations in different fluids and compare the results with GR [26].

4.1 Non-dissipative locally isotropic geodesic fluids

In geodesic fluids, motion of fluid particles are along geodesic so that $a^{\mu} = 0 = a$, which yields $A(t,r) = constant$. Thus for non-dissipative ($q = 0 = \epsilon$), locally isotropic ($\Pi = 0$) and geodesic fluid, we have the following set of evolution equations

$$\Box\phi = \frac{1}{2\omega_{BD}+3}\left[(-\rho+3p_{\perp}) + \left(\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right)\right], \tag{49}$$

$$\left[\rho_{,\mu}u^{\mu} + (\rho+p_r)\Theta\right] + H_3^{\phi} = 0, \tag{50}$$

$$\left[p_{r,\mu}\chi^{\mu}\right] + H_4^{\phi} = 0, \tag{51}$$

$$\Theta_{,\mu}u^{\mu} + \frac{\Theta^2}{3} - \frac{2}{3}\sigma^2 = \frac{1}{\phi}\left[-\frac{1}{2}(\rho+3p_r) - \phi^{;\mu}_{;\nu}u_{\mu}u^{\nu} - \frac{\omega_{BD}}{\phi}\phi^{,\mu}\phi_{,\nu}u_{\mu}u^{\nu} \right.$$
$$\left. + \frac{1}{2}(\Box\phi + V(\phi))\right], \tag{52}$$

$$\left(\frac{\sigma}{2}+\Theta\right)_{,\mu}\chi^{\mu} = -\frac{3\sigma C'}{2BC} - \frac{3A}{2B\phi}\left(\phi_{,t;r} + \frac{\omega_{BD}\phi^{,t}\phi_{,r}}{\phi}\right), \tag{53}$$

$$-\sigma_{,\mu}u^{\mu} + \frac{\sigma^2}{3} - \frac{2\Theta\sigma}{3} = \frac{1}{\phi}\left[\varepsilon + \frac{\Box\phi}{4} - \frac{V(\phi)}{4} - \frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{4\phi} + \frac{\phi_{,t;t}}{4A^2} \right.$$
$$\left. + \frac{\phi_{,r;r}}{4B^2} - \frac{\omega_{BD}}{\phi}\left(\frac{3\dot{\phi}^2}{2A^2} + \frac{\phi'^2}{B^2}\right)\right], \tag{54}$$

$$\left[\left(\varepsilon + \frac{\rho}{2} - \frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{4\phi} + \frac{V(\phi)}{4}\right)_{,\mu}u^{\mu} + \left(\varepsilon + \frac{\rho}{2} + p_r\right)(\Theta+\sigma) \right.$$
$$\left. + \frac{3\omega_{BD}}{2AB}\left(\frac{\phi'\dot{\phi}}{\phi}\right)_{;\mu}\chi^{\mu} + \frac{3}{2B^2}\left(\frac{\phi'^2}{\phi}\right)_{;\mu}u^{\mu}\right] = 0, \tag{55}$$

$$\left(\varepsilon+\frac{\rho}{2}+\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{2\phi}+\frac{\Box\phi}{4}+\frac{V(\phi)}{4}\right)_{,\mu}\chi^{\mu}+\frac{3\omega_{BD}}{2A^{2}}\left(\frac{\dot{\phi}^{2}}{\phi}\right)_{;\alpha}\chi^{\alpha}$$
$$-\frac{3\omega_{BD}}{2AB}\left(\frac{\dot{\phi}\phi'}{\phi}\right)_{;\alpha}u^{\alpha}+\frac{3C'}{BC}(\varepsilon)=0, \tag{56}$$
$$\frac{3m}{C^{3}}=\frac{1}{\phi}\left[\frac{\rho}{2}-\varepsilon+\frac{1}{2}\left[3\Box\phi+\frac{\omega_{BD}\phi_{,c}\phi^{,c}}{\phi}+2V(\phi)\right]\right], \tag{57}$$

where H_3^{ϕ} and H_4^{ϕ} are given in Appendix.

In GR, such fluid represents dust model. Shear free condition ($\sigma=0$) and conformal flatness condition ($\varepsilon=0$) are equivalent, i.e., $\sigma=0$ implies $\varepsilon=0$ and similarly $\varepsilon=0$ gives $\sigma=0$ [26]. In our case, Eq. (51) shows that pressure gradient depends upon scalar field terms with BD coupling constant. If the scalar field has constant distribution, the pressure gradient vanishes yielding dust fluid or equivalently dust condition implies that the scalar field is constant throughout the fluid. Otherwise, fluid distribution has pressure, i.e., the presence of gravitational force due to scalar field makes matter particles to follow intersecting geodesics which induce pressure in the fluid. From Eqs. (50) and (55), it follows that conformal flatness in the fluid depends upon shear as well as scalar field terms (for constant scalar field we have conformal flatness) and similarly, shear-free condition depends upon conformal flatness and scalar field. Equation (52) shows that the evolution of scalar field depends upon density and tangential pressure. Equation (56) indicates that the Weyl tensor and scalar field control energy density inhomogeneity in the fluid. If Weyl tensor vanishes, irregular distribution of scalar field ($\phi\neq constant$) provides density inhomogeneity.

4.2 Non-geodesic fluids

The properties of geodesic fluids are quite different from those in non-geodesic case in GR. Let us see what happens for different models of non-geodesic fluids in BD gravity.

4.2.1 Non-dissipative locally isotropic model

Here we take $\Pi=q=\epsilon=0$ so that the set of governing equations reduce to

$$\Box\phi=\frac{1}{2\omega_{BD}+3}\left[(-\rho+3p_{\perp})+\left(\phi\frac{dV(\phi)}{d\phi}-2V(\phi)\right)\right], \tag{58}$$
$$\left[\rho_{,\mu}u^{\mu}+(\rho+p_r)\Theta\right]+H_1^{\phi}=0, \tag{59}$$
$$\left[p_{r,\mu}\chi^{\mu}+(\rho+p_r)a\right]+H_2^{\phi}=0, \tag{60}$$
$$\Theta_{,\mu}u^{\mu}+\frac{\Theta^{2}}{3}-\frac{2}{3}\sigma^{2}-a^{\mu}_{;\mu}=\frac{1}{\phi}\left[-\frac{1}{2}(\rho+3p_r)-\phi^{;\mu}_{;\nu}u_{\mu}u^{\nu}\right.$$
$$\left.-\frac{\omega_{BD}}{\phi}\phi^{,\mu}\phi_{,\nu}u_{\mu}u^{\nu}+\frac{1}{2}(\Box\phi+V(\phi))\right], \tag{61}$$

$$\left(\frac{\sigma}{2}+\Theta\right)_{,\mu}\chi^{\mu}=-\frac{3\sigma C'}{2BC}-\frac{3A}{2B\phi}\left(\phi_{,t;r}+\frac{\omega_{BD}\phi^{,t}\phi_{,r}}{\phi}\right), \tag{62}$$

$$a_{,\mu}\chi^{\mu}-\sigma_{,\mu}u^{\mu}+a^{2}-\frac{\sigma^{2}}{3}-\frac{2\Theta\sigma}{3}-\frac{aC'}{BC}=\frac{1}{\phi}\left[\varepsilon+\frac{\Box\phi}{4}-\frac{V(\phi)}{4}\right.$$
$$\left.-\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{4\phi}+\frac{\phi_{,t;t}}{4A^{2}}+\frac{\phi_{,r;r}}{4B^{2}}-\frac{\omega_{BD}}{\phi}\left(\frac{3\dot{\phi}^{2}}{2A^{2}}+\frac{\phi'^{2}}{B^{2}}\right)\right], \tag{63}$$

$$\left[\left(\varepsilon+\frac{\rho}{2}-\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{4\phi}+\frac{V(\phi)}{4}\right)_{,\mu}u^{\mu}+\left(\varepsilon+\frac{\rho}{2}+p_{r}\right)(\Theta+\sigma)\right.$$
$$\left.+\frac{3\omega_{BD}}{2AB}\left(\frac{\phi'\dot{\phi}}{\phi}\right)_{;\mu}\chi^{\mu}+\frac{3}{2B^{2}}\left(\frac{\phi'^{2}}{\phi}\right)_{;\mu}u^{\mu}\right]=0, \tag{64}$$

$$\left(\varepsilon+\frac{\rho}{2}+\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{2\phi}+\frac{\Box\phi}{4}+\frac{V(\phi)}{4}\right)_{,\mu}\chi^{\mu}+\frac{3\omega_{BD}}{2A^{2}}\left(\frac{\dot{\phi}^{2}}{\phi}\right)_{;\alpha}\chi^{\alpha}$$
$$-\frac{3\omega_{BD}}{2AB}\left(\frac{\dot{\phi}\phi'}{\phi}\right)_{;\alpha}u^{\alpha}+\frac{3C'}{BC}(\varepsilon)=0, \tag{65}$$

$$\frac{3m}{C^{3}}=\frac{1}{\phi}\left[\frac{\rho}{2}-\varepsilon+\frac{1}{2}\left[3\Box\phi+\frac{\omega_{BD}\phi_{,c}\phi^{,c}}{\phi}+2V(\phi)\right]\right]. \tag{66}$$

In GR, for this type of fluid, density homogeneity gives conformal flatness or equivalently conformal flatness provides energy density homogeneity. Moreover, shear-free condition implies expansion free fluid. From Eq. (65), we have

$$\left(C^{3}\varepsilon\right)_{,\mu}\chi^{\mu}+C^{3}\left(\frac{\rho}{2}\right)_{,\mu}\chi^{\mu}=\left(-\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{2\phi}-\frac{\Box\phi}{4}-\frac{V(\phi)}{4}\right)_{,\mu}\chi^{\mu}$$
$$-\frac{3\omega_{BD}}{2A^{2}}\left(\frac{\dot{\phi}^{2}}{\phi}\right)_{;\alpha}\chi^{\alpha}+\frac{3\omega_{BD}}{2AB}\left(\frac{\dot{\phi}\phi'}{\phi}\right)_{;\alpha}u^{\alpha}-\frac{3C'}{BC}(\varepsilon), \tag{67}$$

which shows that energy density inhomogeneity not only depends upon the Weyl tensor but also on energy terms due to scalar field. If the fluid is conformally flat then density inhomogeneity is totally due to scalar field. Also, it follows from Eq. (62) that shear free condition does not imply expansion free fluid due to the presence of scalar field.

4.2.2 Non dissipative locally anisotropic model

In this case, we have $q=\epsilon=0$ but $\Pi\neq 0$. Consequently Eq. (65), yields

$$\left(C^{3}(\varepsilon+\frac{\Pi}{2})\right)_{,\mu}\chi^{\mu}+C^{3}\left(\frac{\rho}{2}\right)_{,\mu}\chi^{\mu}=\left(-\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{2\phi}-\frac{\Box\phi}{4}-\frac{V(\phi)}{4}\right)_{,\mu}\chi^{\mu}$$
$$-\frac{3\omega_{BD}}{2A^{2}}\left(\frac{\dot{\phi}^{2}}{\phi}\right)_{;\alpha}\chi^{\alpha}+\frac{3\omega_{BD}}{2AB}\left(\frac{\dot{\phi}\phi'}{\phi}\right)_{;\alpha}u^{\alpha}-\frac{3C'}{BC}(\varepsilon), \tag{68}$$

which shows that energy density inhomogeneity depends upon anisotropy, Weyl tensor and energy terms due to scalar field. In GR, this type of fluid relates density inhomogeneity with anisotropy and Weyl tensor.

4.2.3 Dissipative locally anisotropic model

Here we take $\tilde{q}\neq 0,\ \Pi\neq 0$, so Eq. (65) implies that

$$\begin{aligned}\left(C^3\left(\varepsilon+\frac{\Pi}{2}\right)\right)_{,\mu}\chi^{\mu}+C^3\left(\frac{\tilde{\rho}}{2}\right)_{,\mu}\chi^{\mu}&=\left(-\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{2\phi}-\frac{\Box\phi}{4}-\frac{V(\phi)}{4}\right)_{,\mu}\chi^{\mu}\\&\quad-\frac{3\omega_{BD}}{2A^2}\left(\frac{\dot{\phi}^2}{\phi}\right)_{;\alpha}\chi^{\alpha}+\frac{3\omega_{BD}}{2AB}\left(\frac{\dot{\phi}\phi'}{\phi}\right)_{;\alpha}u^{\alpha}-\frac{3C'}{BC}(\varepsilon)-\tilde{q}(\sigma+\Theta).\end{aligned}\tag{69}$$

This represents the roles of Weyl tensor, anisotropy, dissipation and scalar field in the inhomogeneity of energy density. If there are no expansion and shear in the fluid then the dissipation does not affect the density inhomogeneity. Moreover, if the scalar field is regular inside the fluid (constant), the energy inhomogeneity does not depend upon the scalar field.

To discuss, the role of thermodynamics in the density inhomogeneity, we consider the transport equation [21–25]

$$\tau h^{\alpha}_{\beta}(q^{\beta})_{,\mu}u^{\mu}+q^{\alpha}=\kappa h^{\alpha\beta}(T_{,\beta}+Ta_{\beta})-\frac{1}{2}\kappa\left(\frac{\tau u^{\beta}}{\kappa T^2}\right)_{;\beta}q^{\alpha},\tag{70}$$

where T, κ and τ represent temperature, thermal conductivity and relaxation time, respectively. Using Eq. (29), (70) and condition $\epsilon=0$ (for simplicity), we have

$$q=\frac{\tau\left[p_{r,\mu}\chi^{\mu}+(\rho+p_r)a+\Pi\frac{2C'}{CB}++H_2^{\phi}\right]-\kappa\left(T_{\mu}u^{\mu}+Ta\right)}{1+\frac{\tau}{2}\left[\frac{1}{3}(2\sigma-5\Theta)+\frac{\tau_{,\mu}u^{\mu}}{\tau}-\frac{\kappa_{,\mu}u^{\mu}}{\kappa}-\frac{2T_{,\mu}u^{\mu}}{T}\right]}.\tag{71}$$

Substituting Eq. (71) in (69), it follows that

$$\begin{aligned}\left(C^3\left(\varepsilon+\frac{\Pi}{2}\right)\right)_{,\mu}\chi^{\mu}+C^3\left(\frac{\rho}{2}\right)_{,\mu}\chi^{\mu}&=\left(-\frac{\omega_{BD}\phi^{,\mu}\phi_{,\mu}}{2\phi}-\frac{\Box\phi}{4}-\frac{V(\phi)}{4}\right)_{,\mu}\chi^{\mu}\\&\quad-\frac{3\omega_{BD}}{2A^2}\left(\frac{\dot{\phi}^2}{\phi}\right)_{;\alpha}\chi^{\alpha}+\frac{3\omega_{BD}}{2AB}\left(\frac{\dot{\phi}\phi'}{\phi}\right)_{;\alpha}u^{\alpha}-\frac{3C'}{CB}(\varepsilon)\\&\quad-\frac{\tau\left[p_{r,\mu}\chi^{\mu}+(\rho+p_r)a+\Pi\frac{2C'}{CB}++H_2^{\phi}\right]-\kappa\left(T_{\mu}u^{\mu}+Ta\right)}{1+\frac{\tau}{2}\left[\frac{1}{3}(2\sigma-5\Theta)+\frac{\tau_{,\mu}u^{\mu}}{\tau}-\frac{\kappa_{,\mu}u^{\mu}}{\kappa}-\frac{2T_{,\mu}u^{\mu}}{T}\right]}(\sigma+\Theta).\end{aligned}\tag{72}$$

This shows a relation between thermodynamics and energy density inhomogeneity.

5 Conclusions

In this paper, we have developed a set of governing equations for spherically symmetric self-gravitating fluid models in BD gravity. These equations explain the role of dynamical variables as well as scalar field in the evolution of self-gravitating fluids. We have discussed different fluid models by taking different dynamical conditions. It is found that gravitational effects due to the presence of scalar field play an important role in the dynamics of evolving self-gravitating fluids. We can summarize our main results and compare with GR [26] as follows.

- For non-dissipative perfect (locally isotropic) geodesic fluids, the fluid distribution is dust if the scalar field is constant inside the fluid, otherwise pressure is induced by the gravitational force due to scalar field. Moreover, the conformal flatness is not obtained from shear-free condition and energy density inhomogeneity depends on the Weyl tensor as well as scalar field. In GR, this case represents dust models in which shear-free condition implies conformal flatness and energy inhomogeneity depends upon the Weyl tensor.
- In non-dissipative locally isotropic and non-geodesic fluids, density inhomogeneity is due to gravitational force mediated by Weyl tensor as well as scalar field and shear free condition does not imply expansion free fluid. According to GR, energy density homogeneity is due to conformal flatness while shear-free condition implies expansion free fluid.
- For non-dissipative anisotropic fluid, energy inhomogeneity involves scalar field along with other dynamical variables while in GR it is due to the presence of Weyl tensor and anisotropic pressure.
- In the case of dissipative anisotropic fluid, the evolution takes place due to gravitational force of scalar field along with other dynamical variables. In GR, the evolution is affected by the relation among different dynamical variables (Weyl tensor, anisotropy and dissipation).

Finally, we conclude that in BD theory, all dynamical changes of evolving fluid depend upon the behavior of scalar field rather than geometrical properties (Weyl tensor, shear tensor etc). For regular (or constant) scalar field inside the fluid, the resulting models possess dust phase, density homogeneity, conformally flatness and shear-free conditions, which are same as defined in GR [26]. Otherwise, irregular distribution of scalar field preserves dynamical properties (density inhomogeneity, pressure as well as its anisotropic effects, dissipation, tidal forces ($\varepsilon \neq 0$) and effects of shear force) which yields a deviation from GR. Thus the corresponding fluid models are would be more realistic as compared to GR.

6 Appendix

The scalar field terms of Eqs. (28) and (29) are given by

$$H_1^{\phi} = -A\left(\phi^{,\mu;\nu} + \frac{\omega_{BD}\phi^{,\mu}\phi^{,\nu}}{\phi}\right)_{;\nu} + \left(\frac{2\phi\Box\phi + \omega_{BD}\phi^{,\alpha}\phi_{,\alpha} - \phi V(\phi)}{2A\phi}\right)_{;\nu} + \frac{\dot{\phi}}{\phi}$$

$$\times\left(-A\tilde{\rho}+\phi^{,t;t}+\frac{\omega_{BD}(\phi^{,t})^2}{\phi}-\frac{\omega_{BD\phi^{,\alpha}\phi_{,\alpha}}}{2\phi A}\right)$$
$$+\frac{\phi'}{\phi}\left(\tilde{q}AB+\phi^{,t;r}+\frac{\omega_{BD}\phi^{,t}\phi^{,r}}{\phi}\right),$$
$$H_2^{\phi}=B\left(\phi^{,\mu;\nu}+\frac{\omega_{BD}\phi^{,\mu}\phi^{,\nu}}{\phi}\right)_{;\nu}+\left(\frac{2\phi\Box\phi+\omega_{BD}\phi^{,\alpha}\phi_{,\alpha}-\phi V(\phi)}{2B\phi}\right)_{;\nu}+\frac{\dot{\phi}}{\phi}$$
$$\times\left(\tilde{q}AB+\phi^{,t;r}+\frac{\omega_{BD}\phi^{,t}\phi^{,r}}{\phi}\right)$$
$$+\frac{\phi'}{\phi}\left(\tilde{p}_r+\phi^{,r;r}+\frac{\omega_{BD}(\phi^{,r})^2}{\phi}-\frac{\omega_{BD\phi^{,\alpha}\phi_{,\alpha}}}{2\phi B}\right).$$

The scalar energy terms H_3^{ϕ} and H_4^{ϕ} in geodesic fluid are as follows

$$H_3^{\phi}=-A\left(\phi^{,\mu;\nu}+\frac{\omega_{BD}\phi^{,\mu}\phi^{,\nu}}{\phi}\right)_{;\nu}+\left(\frac{2\phi\Box\phi+\omega_{BD}\phi^{,\alpha}\phi_{,\alpha}-\phi V(\phi)}{2A\phi}\right)_{;\nu}+\frac{\dot{\phi}}{\phi}$$
$$\left(-A\tilde{\rho}+\phi^{,t;t}+\frac{\omega_{BD}(\phi^{,t})^2}{\phi}-\frac{\omega_{BD\phi^{,\alpha}\phi^{,\alpha}}}{2\phi A}\right)+\frac{\phi'}{\phi}\left(\phi^{,t;r}+\frac{\omega_{BD}\phi^{,t}\phi^{,r}}{\phi}\right),$$
$$H_4^{\phi}=B\left(\phi^{,\mu;\nu}+\frac{\omega_{BD}\phi^{,\mu}\phi^{,\nu}}{\phi}\right)_{;\nu}+\left(\frac{2\phi\Box\phi+\omega_{BD}\phi^{,\alpha}\phi_{,\alpha}-\phi V(\phi)}{2B\phi}\right)_{;\nu}+\frac{\dot{\phi}}{\phi}$$
$$\left(\phi^{,t;r}+\frac{\omega_{BD}\phi^{,t}\phi^{,r}}{\phi}\right)+\frac{\phi'}{\phi}\left(\tilde{p}_r+\phi^{,r;r}+\frac{\omega_{BD}(\phi^{,r})^2}{\phi}-\frac{\omega_{BD\phi^{,\alpha}\phi_{,\alpha}}}{2\phi B}\right).$$

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