Dynamical Analysis of Charged Anisotropic Spherical Star in $f(R)$ Gravity

H. Rizwana Kausar [*(a)], Ifra Noureen [†(b)], M. Umair Shahzad [‡(a)]
[(a)] CAMS, UCP Business School,
University of Central Punjab, Lahore, Pakistan.
[(b)] Department of Mathematics,
University of Management and Technology,
Lahore, Pakistan.

Abstract

We consider a modified gravity theory, $f(R) = R + \alpha R^n - \frac{\mu^4}{R^m}$, in the metric formulation and analyze the contribution of electromagnetic field on the range of dynamical instability of a star filled with anisotropic matter. The collapse equation is developed by applying conservation on anisotropic matter, Maxwell source and dark source terms arising due to $f(R)$ gravity. Specific perturbation scheme is implemented and it is observed that the inclusion of Maxwell source slows down the collapse and makes system more stable in Newtonian regime. Also, we make comparison of our results with the existing literature.

1 Introduction

The investigations about the final outcome of a stellar collapse and stability of a self-gravitating body is always a key issue in astrophysics and gravitational

*rizwa_math@yahoo.com
†ifra.noureen@gmail.com
‡m.u.shahzad@ucp.edu.pk

theories. The astrophysical bodies remain stable until the outward pressure produced by internal fusion balances the inward acting gravity force. The endless collapsing phenomenon takes place when a massive star exhaust its nuclear fuel, the gravitational force then take over implying the death of a star [1]. The life cycle of a massive star having mass of the order $10-20$ solar mass is nominal in comparison to the stars with relatively less mass i.e., ≈ 1 solar mass. During collapse, the energy dissipation in the form of heat flow produces different scenarios for different sizes of stars [2]-[4]. A more massive star tends to be more unstable due to high radiation transport rather than a less massive star.

The self-gravitating object is worthwhile only if it is stable against fluctuations. The pioneer contribution on the problem of dynamical instability is by Chandrasekhar [5], who found instability range in the form of adiabatic index Γ as $\Gamma \geq \frac{4}{3} + n\frac{M}{r}$, where M and R are the mass and radius of a star respectively. Hillebrandt and K.O. Steinmetz [6] establish the stability criterion for the model of anisotropic neutron stars.

Herrera and his contemporaries established the stability ranges for various forms of gravitational collapse by considering isotropy, local anisotropy, expansion free condition, dissipation etc. [7]-[9]. It was established that minor changes in fluid isotropy affects subsequent evolution considerably, also, in relativistic annexes the dissipative, radiative and shearing effects gave rise to the stability of compact object [10]-[14] by slowing down the collapse.

The collapse and stability of gravitating bodies in alternative gravity theories have been studied extensively. Among modified theories, $f(R)$ represents the most simple alteration in Einstein-Hilbert (EH) action to include higher order curvature invariants by replacing Ricci scalar R with the terms of order $\sqrt{-g}f(R)$. Here $f(R)$ denote the general function of R and g stands for the metric tensor [15]-[17]. Thus the action for $f(R)$ gravity can be written as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R),$$

where κ is the coupling constant. The viability and stability analysis in context of modified theories of gravity has been presented in [18]-[30] in view of observational situations such as cosmic microwave background, clustering spectrum and weak lensing [31]-[34].

The electromagnetic field affects the evolution and structure stability range of the collapsing relativistic bodies. The magnetic and Coulomb forces are the major source for resisting the gravitational pull [35]. A star needs a large amount of charge in order to stay stable among the strong gravitational pull [36]. Such electromagnetic tension also leads to homogeneity on large scales. In modified $f(R)$ theory, some work has been done on the aspect of instability of collapsing stars by considering combined effect of electromagnetic field and $f(R)$ model [36]-[39]. Motivating by this work, we have chosen $f(R) = R + \alpha R^n - \frac{\mu^4}{R^m}$ and charged anisotropic fluid to workout the instability problem. The matching of stable stellar configuration corresponds to positive real values of α, m and n, while μ can be any real number. However, to meet viability criteria of the proposed model, we need to fix $n \geq 2, m \geq 1$ to keep $\frac{d^2 f}{dR^2}$ positive.

The manuscript is arranged in the following manner: Section **2** provides energy-momentum tensor for usual matter, Maxwell source and dark source along with Einstein's field equations and dynamical equations. In section **3**, the perturbation scheme is presented and discussions on instability in terms of adiabatic index is given. Summary of the obtained results is provided in section **4** which is followed by an appendix.

2 Evolution Equations

Here, a three dimensional timelike spherical boundary Σ is chosen that imparts space time into two regions referred as interior and exterior regions respectively. The line element for region inside the spherical boundary surface Σ is given by

$$ds_-^2 = A^2(t,r)dt^2 - B^2(t,r)dr^2 - C^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

The exterior region is defined by the line element as follow [39]

$$ds_+^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) d\nu^2 + 2drd\nu - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.2}$$

where Q stands for the total charge, ν is the retarded time and M denotes the total mass. The EH action in $f(R)$ theory including Maxwell source is defined as

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(\frac{f(R)}{\kappa} - \frac{F}{2\pi}\right), \tag{2.3}$$

where $F = \frac{1}{4}F^{uv}F_{uv}$ is the Maxwell invariant. By varying above action with respect to the metric tensor g_{uv}, we obtain the following set of field equations

$$f_R R_{uv} - \frac{1}{2}f(R)g_{uv} - \nabla_u\nabla_v f_R + g_{uv}\Box f_R = \kappa(T_{uv} + E_{uv}), \quad (u, v = 0, 1, 2, 3). \tag{2.4}$$

Here $f_R \equiv df(R)/dR$, $\Box = \nabla^u\nabla_v$ where ∇_u represents the covariant derivative, E_{uv} is the electromagnetic tensor and T_{uv} is the minimally coupled stress-energy tensor. Rewriting, above set of field equations in the form of Einstein tensor on the left hand side, we have

$$G_{uv} = \frac{\kappa}{f_R}[L_{uv}], \quad \text{where} \quad L_{uv} = \overset{(D)}{T}_{uv} + T_{uv} + E_{uv}. \tag{2.5}$$

For interior spherically symmetric metric, the components of Einstein tensor can be found in [39]. In the above equation $\overset{(D)}{T}_{uv}$ is supposed to known as dark source energy contribution due to modification in gravity. It is given by

$$\overset{(D)}{T}_{uv} = \frac{1}{\kappa}\left[\frac{f(R) - Rf_R}{2}g_{uv} + \nabla_u\nabla_v f_R - g_{uv}\Box f_R\right], \tag{2.6}$$

The energy momentum tensor T_{uv} is referred to the usual matter under consideration which is assumed to be charged anisotropic fluid dissipating energy via heat flow q. It is defined as

$$T_{uv} = (\mu + p_\perp)V_uV_v - p_\perp g_{uv} + (p_r - p_\perp)\chi_u\chi_v + q_uV_v + q_vV_u, \tag{2.7}$$

where V_u is the for four-velocity, μ is the density, p_r and $p_\perp$ corresponds to redial and tangential pressures respectively and χ_u is the corresponding radial four vector. These quantities satisfy the following relations in co-moving coordinates

$$V^u = A^{-1}\delta^u_0, \quad q^u = qB^{-1}\delta^u_1, \quad \chi^u\chi_u = -1, \quad \chi^u = B^{-1}\delta^u_1. \tag{2.8}$$

The energy momentum tensor corresponding to Maxwell source is given by [39]

$$E_{uv} = \frac{1}{4\pi}(-F^w_u F_{vw} + \frac{1}{4}F^{wx}F_{wx}g_{uv}), \tag{2.9}$$

where electromagnetic field tensor F_{uv} is defined by $F_{uv} = \varphi_{v,u} - \varphi_{u,v}$ with $\varphi_u = \varphi(t, r)\delta^0_u$ representing the four potential. The Maxwell's equations are [40]

$$F^{uv}_{;v} = 4\pi\jmath^u, \quad F_{uv;w} = 0, \tag{2.10}$$

where four current is $\jmath^u = \xi(t,r)V^u$ and ξ stands for charge density. On simplification, electromagnetic field equations become

$$\frac{\partial^2\varphi}{\partial r^2} - \left(\frac{A'}{A} + \frac{B'}{B} - \frac{2C'}{C}\right)\frac{\partial\varphi}{\partial r} = 4\pi\mu AB^2, \tag{2.11}$$

$$\frac{\partial^2\varphi}{\partial t\partial r} - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{2\dot{C}}{C}\right)\frac{\partial\varphi}{\partial r} = 0. \tag{2.12}$$

Herein dot and prime indicate time and radial derivatives respectively. The charge conservation is defined as

$$s(r) = \int_0^r \mu BC^2 dr, \quad E = \frac{s}{4\pi C^2}, \tag{2.13}$$

where time independent charge interior to radius r is denoted by s and E is the electric field intensity. Integrating Eq.(2.11), we get

$$\frac{\partial\varphi}{\partial r} = \frac{sBA}{C^2}. \tag{2.14}$$

The matching conditions across boundary surface are of great interest in order to seek the valid solution to the field equations for both the interior and exterior metrics. The smooth matching conditions corresponding to Darmois junction conditions [42] imply that the induced metric and extrinsic curvature must be continuous on both sides of the hypersurface. Relativistic objects bearing discontinuity in the extrinsic curvatures can be described by the presence of a thin-shell formalism presented by Israel [43, 44]. Matching of interior and exterior spacetimes across Σ yield

$$m(t,r) \overset{\Sigma}{=} M \Longleftrightarrow q(r) \overset{\Sigma}{=} Q, \quad p_\perp \overset{\Sigma}{=} \frac{-1}{B^2}\left(\frac{B}{A}\overset{(D)}{T}_{01} + \overset{(D)}{T}_{11}\right),$$

where $m(t,r)$ denotes the Misner-Sharp mass given by [45]

$$m(t,r) = \frac{C}{2}\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right) + \frac{q^2}{2C}.$$

The Bianchi identities/dynamical equations are employed to workout the evolution of gravitating objects by applying conservation on usual matter along with dark source terms and electromagnetic energy momentum tensor as

$$L^{uv}_{;v}V_u = 0, \quad L^{uv}_{;v}\chi_u = 0. \tag{2.15}$$

Consequently, we obtain following two equations

$$\begin{aligned}
&\dot{\rho} + q'\frac{A}{B} + 2q\frac{A}{B}\left(\frac{A'}{A} + \frac{C'}{C}\right) + \rho\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + p_r\frac{\dot{B}}{B} + 2p_\perp\frac{\dot{C}}{C} \\
&+P_1(r,t) = 0, \qquad (2.16)\\
&p'_r + p_r\left(\frac{A'}{A} + \frac{C'}{C}\right) + \rho\frac{A'}{A} - 2p_\perp\frac{C'}{C} + \dot{q}\frac{B}{A} + 2\frac{B}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \\
&-4\pi E^2\left(\frac{E'}{E} + \frac{2C'}{C}\right) + P_2(r,t) = 0. \qquad (2.17)
\end{aligned}$$

The dark matter terms in $P_1(r,t)$ and $P_2(r,t)$ are given in **Appendix A** as Eqs.(4.1) and (4.2) respectively.

3 The Perturbed Dynamical Equations

The perturbation scheme is being employed here to acquire the simplified form of non-linear field equations whose general solution has not been established yet. The perturbed form of dynamical equations lead to the linear equations in the form of material and metric variables. The dynamics of evolution can be investigated either by following Eulerian or Lagrangian pattern, i.e., fixed or co-moving coordinates respectively. Here we have chosen co-moving coordinates in which it is assumed that all the quantities are in static equilibrium initially. However, with the passage of time these quantities have small perturbations in the form of ϵ and acquire time dependence too. In this approach, our chosen $f(R)$ model which is

$$f(R) = R + \alpha R^n - \frac{\mu^4}{R^m} \tag{3.1}$$

would also be perturbed along the same pattern. Thus, we can write all the quantities in their perturbed form as

$$
\begin{aligned}
A(t,r) &= A_0(r)+\epsilon T(t)a(r), && (3.2)\\
B(t,r) &= B_0(r)+\epsilon T(t)b(r), && (3.3)\\
C(t,r) &= r+\epsilon T(t)\bar{c}(r), && (3.4)\\
\rho(t,r) &= \rho_0(r)+\epsilon\bar{\rho}(t,r), && (3.5)\\
p_r(t,r) &= p_{r0}(r)+\epsilon\bar{p}_r(t,r), && (3.6)\\
p_\perp(t,r) &= p_{\perp 0}(r)+\epsilon\bar{p}_\perp(t,r), && (3.7)\\
q(t,r) &= \epsilon\bar{q}(t,r), && (3.8)\\
E(t,r) &= E_0(r)+\epsilon T(t)h(r), && (3.9)\\
R(t,r) &= R_0(r)+\epsilon T(t)e(r), && (3.10)\\
f(R) &= \left(R_0+\alpha R_0^n-\mu^4R_0^{-m}\right)+\epsilon T(t)e(r)\left(1+\alpha nR_0^{n-1}\right.\\
&+ \left.\mu^4mR_0^{-m-1}\right), && (3.11)\\
f_R(R) &= \left(1+\alpha nR_0^{n-1}+\mu^4mR_0^{-m-1}\right)+\epsilon T(t)e(r)\left[\alpha n(n-1)R_0^{n-2}\right.\\
&- \left.\mu^4m(m+1)R_0^{-m-2}\right]. && (3.12)
\end{aligned}
$$

Applying above set of equations on Bianchi identities (2.16) and (2.17), we have

$$
\begin{aligned}
&\frac{\bar{\rho}}{A_0^2}+\frac{2\bar{q}A_0'}{A_0^2B_0}+\frac{\bar{q}'}{A_0B_0}+\frac{2\bar{q}}{rA_0B_0}+\left[\frac{b(p_{r0}+\rho_0)}{A_0^2B_0^2}\right.\\
&\left.+\frac{2\bar{c}(p_{\perp 0}-\rho_0)}{rA_0^2}+P_{1p}\right]\dot{T}=0, && (3.13)\\
&\frac{\dot{\bar{\rho}}}{A_0B_0}+\frac{\bar{p_r}'}{B_0^2}+\frac{A_0'}{A_0B_0^2}(\bar{\rho}+\bar{p_r})+\frac{2}{rB_0^2}(\bar{p_r}-\bar{p_\perp})+\left[\frac{A_0'}{A_0B_0^2}(\rho_0\right.\\
&+p_{r0})\left(\frac{a'}{A_0'}-\frac{a}{A_0}-\frac{2b}{B_0}\right)+\frac{2}{rB_0^2}(p_{r0}-p_{\perp 0})\left(\bar{c}'-\frac{\bar{c}}{r}-\frac{2b}{B_0}\right)\\
&\left.-4\pi\left\{(E_0h)'+2E_0^2\left(\frac{\bar{c}}{r}\right)'+2E_0h(\frac{E_0'}{E_0}+\frac{2}{r})\right\}+P_{2p}\right]T=0. && (3.14)
\end{aligned}
$$

The perturbed configurations of P_1 and P_2 are denoted by P_{1p} and P_{2p} respectively and are provided in the **Appendix B**, Eqs.(4.3)-(4.4). Heat flux can be found by applying perturbation on the field equation $G_{01}=L_{01}$. The obtained relation is given as follow

$$\begin{aligned}
\bar{q} &= \left[\frac{\alpha R_0^{n-1}}{\kappa A_0 B_0}\left\{(n-1)R_0^{-1}+(n-1)(n-2)R_0^{-2}R_0' e-(n-1)R_0^{-1}\left(\frac{eA_0'}{A_0}\right.\right.\right.\\
&\left.\left.+\frac{b}{B_0}\right)+2\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\right\}+\frac{\mu^4 m R_0^{-m-1}}{\kappa A_0 B_0}\{-(m+1)R_0^{-1}e'\\
&+(m+1)(m+2)R_0^{-2}R_0' e+(m+1)R_0^{-1}\left(\frac{eA_0'}{A_0}+\frac{b}{B_0}\right)+2\left(\frac{\bar{c}'}{r}\right.\\
&\left.\left.-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\right\}+\left.\frac{2}{\kappa A_0 B_0}\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\right]\dot{T}. \qquad (3.15)
\end{aligned}$$

By substituting above value of $\bar{q}$ and its derivative in Eq.(3.13), we obtain

$$\bar{\rho} = -\left[(p_{r0}+\rho_0)\frac{b}{B_0^2}+(p_{\perp 0}-\rho_0)\frac{2\bar{c}}{r}+P_3\right]T. \qquad (3.16)$$

Here expression for P_3 constitutes the dark source terms of the proposed $f(R)$ model and is given in the **Appendix B**. The Harrison-Wheeler type equation of state describes second law of thermodynamics, that associates $\bar{\rho}$ and $\bar{p_r}$ in the following pattern

$$\bar{p_r} = \Gamma\frac{p_{r0}}{\rho_0+p_{r0}}\bar{\rho}, \qquad (3.17)$$

where Γ is the adiabatic index that amounts the pressure fluctuation with density variation. Insertion of Eq.(4.5) in the above equation, it implies

$$\bar{p_r} = -\Gamma\left[p_{r0}\frac{b}{B_0^2}+p_{r0}\left(\frac{p_{\perp 0}-\rho_0}{p_{ro}+\rho_0}\right)\frac{2\bar{c}}{r}+\frac{p_{r0}}{p_{r0}+\rho_0}P_3\right]T. \qquad (3.18)$$

Substituting $\bar{q}$, $\bar{\rho}$, and $\bar{p}_r$ from Eqs.(3.15), (3.16) and (3.18) respectively in (3.14), we obtain the following required collapse equation

$$
\begin{aligned}
&-\Gamma\left[p_{r0}\frac{b}{B_0^2}+p_{r0}\left(\frac{p_{\perp0}-\rho_0}{\rho_0+p_{r0}}\right)\frac{2\bar{c}}{r}+\frac{p_{r0}}{p_{r0}+\rho_0}\left\{P_3+4\pi E_0^2\left(\frac{h}{E_0}\right.\right.\right.\\
&\left.\left.\left.+\frac{2\bar{c}}{r}\right)\right\}\right]'T-\left[\frac{A_0'}{A_0}+\Gamma\frac{p_{r0}}{\rho_0+p_{r0}}\left(\frac{A_0'}{A_0}+\frac{2}{r}\right)\right]\left[(\rho_0+p_{r0})\frac{b}{B_0^2}\right.\\
&\left.+(p_{\perp0}-\rho_0)\frac{2\bar{c}}{r}+P_3+4\pi E_0^2\left(\frac{h}{E_0}+\frac{2\bar{c}}{r}\right)\right]T-\frac{2}{r}p_\perp+\left[\frac{-2bp_{ro}'}{B_0}\right.\\
&+\frac{A_0'}{A_0}(\rho_0+p_{r0})\left(\frac{a'}{A_0'}-\frac{a}{A_0}-\frac{2b}{B_0}\right)+\frac{2}{r}(p_{r0}-p_{\perp0})\left(\bar{c}-\frac{\bar{c}}{e}-\frac{2b}{B_0}\right)\\
&\left.-4\pi\left\{(E_0h)'+2E_0^2(\frac{\bar{c}}{r})'+2E_0h(\frac{E_0'}{E_0}+\frac{2}{r})\right\}+P_{2p}\right]T+\left[\frac{\alpha nR_0^{n-1}}{\kappa A_0^2}\right.\\
&\times\left[(n-1)R_0^{-1}e'+(n-1)(n-2)R_0^{-2}R_0'e-(n-1)R_0'\left(e\frac{A_0'}{A_0}+\frac{b}{B_0}\right)\right.\\
&\left.\left.+2\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\right]+\frac{\mu^4mR_0^{-m-1}}{\kappa A_0B_0}\left[-(m+1)R_0^{-1}e'+(m\right.\right.\\
&+1)(m+2)R_0^{-2}R_0'e+(m+1)R_0^{-1}\left(\frac{eA_0'}{A_0}+\frac{b}{B_0}\right)+2\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}\right.\\
&\left.\left.\left.-\frac{b}{rB_0}\right)\right]+\frac{2}{\kappa A_0B_0}\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\right]\ddot{T}=0. \qquad (3.19)
\end{aligned}
$$

This equations helps to study the evolution of spherical charged collapsing star. The perturbed configuration of Ricci scalar yields an ordinary differential equation in the form below

$$\ddot{T}(t)-P_4(r)T(t)=0, \qquad (3.20)$$

where P_4 is provided in the **Appendix B**, Eq.(4.6). In order to establish instability range, the terms appearing in P_4 are assumed in such a way that over all expression should be positive. The solution of Eq.(3.20) is given by

$$T(t)=-e^{\sqrt{P_4}t}. \qquad (3.21)$$

Newtonian Limit

To approximate our results in Newtonian gravity, we assume and substitute $\rho_0 \gg p_{r0},\ \rho_0 \gg p_{\perp0}$ and $A_0=1,\ B_0=1$ in Eq.(3.19) and get collapse equation as

$$\Big[\Gamma\left(-p_{r0}b\right)'T-\frac{2}{r}p_{\perp}\left(\frac{1}{T}\right)-2bp'_{ro}+a'\rho_0+\frac{2}{r}(p_{r0}-p_{\perp0})\left(\bar{c}'-\frac{\bar{c}}{r}-2b\right)$$
$$+P^{N}_{2P}\Big]T+\left[\frac{\alpha nR_0^{n-1}}{\kappa}\left[(n-1)R_0^{-1}e'+(n-1)(n-2)R_0^{-2}R'_0e-(n-1)R'_0b\right.\right.$$
$$\left.\left.+2\left(\frac{\bar{c}'}{r}-\frac{b}{r}\right)\right]+\frac{\mu^4mR_0^{-m-1}}{\kappa}\left[-(m+1)R_0^{-1}e'+(m+1)(m+2)R_0^{-2}R'_0e\right.\right.$$
$$\left.\left.+(m+1)R_0^{-1}b\right]\frac{2}{\kappa}\left(\frac{\bar{c}'}{r}-\frac{b}{r}\right)\right]\ddot{T}-4\pi\left\{(E_0h)'+2E_0^2(\frac{\bar{c}}{r})'+2E_0h(\frac{E'_0}{E_0}\right.$$
$$\left.+\frac{2}{r})\right\}=0. \tag{3.22}$$

Here, P^N_{2p} represents the Newtonian approximation of perturbed second Bianchi identity. Substituting value of T from Eq.(3.21) and its subsequent time derivatives in the above equation, we get

$$\Gamma<\frac{a'\rho_0-2bp'_{ro}+\frac{2}{r}(p_{r0}-p_{\perp0})\left(\bar{c}'-\frac{\bar{c}}{r}-2b\right)+E_1+P^N_{2p}+P_5(r)}{\left(-p_{r0}b\right)'} \tag{3.23}$$

where $E_1=-4\pi\left\{(E_0h)'+2E_0^2(\frac{\bar{c}}{r})'+2E_0h(\frac{E'_0}{E_0}+\frac{2}{r})\right\}$ and P_5 is given in the **Appendix B**. The terms involving in E_1 depicts that the resistance offered to the inward drawn gravity include electromagnetic field components detaining collapse and hence imply a wider range of stability. The expression Γ have various limiting cases in the Newtonian regime as follows:

1. when $\mu\rightarrow0$, the expression for Γ will remains the same for usual matter, however, effective part P_5 reduced to P_6 as

$$\begin{aligned}P_6(r)&=&P_4\left[\frac{\alpha nR_0^{n-1}}{\kappa}\left[(n-1)R_0^{-1}e'+(n-1)(n-2)R_0^{-2}R'_0e-(n\right.\right.\\&&\left.\left.-1)R'_0b+2\left(\frac{\bar{c}'}{r}-\frac{b}{r}\right)\right]+\frac{2}{\kappa}\left(\frac{\bar{c}'}{r}-\frac{b}{r}\right)\right].\end{aligned} \tag{3.24}$$

2. when $\alpha\rightarrow0$, P_4 changes to P_7, whereas terms involving matter variables remain unchanged. The expression for P_7 is given by

$$\begin{aligned}P_7(r)&=&P_4\left[\frac{\mu^4mR_0^{-m-1}}{\kappa}\left[-(m+1)R_0^{-1}e'+(m+1)(m+2)R_0^{-2}R'_0e\right.\right.\\&&\left.\left.+(m+1)R_0^{-1}b\right]+\frac{2}{\kappa}\left(\frac{\bar{c}'}{r}-\frac{b}{r}\right)\right].\end{aligned} \tag{3.25}$$

3. When both $\alpha \to 0$ and $\mu \to 0$, Γ becomes

$$\Gamma < \frac{a'\rho_0 - 2bp'_{ro} + \frac{2}{r}(p_{r0} - p_{\perp 0} + \frac{p_\perp}{r})\left(\bar{c}' - \frac{\bar{c}}{r} - 2b\right) + E_1 + \frac{P_4(r)}{\kappa}\left(\frac{2\bar{c}'}{r} - \frac{2b}{r}\right)}{\left(-p_{r0}b\right)'} \tag{3.26}$$

representing the GR case.

It is mentioned that the above mentioned limiting cases are consistent with the previously done work on $f(R)$ models. Also, for the post Newtonian limit, the relativistic effects can be analyzed upto $O(\frac{m_0}{r} + \frac{Q^2}{2r^2})$ by assuming metric coefficients as

$$A_0 = 1 - \frac{m_0}{r} + \frac{Q^2}{2r^2}, \quad B_0 = 1 + \frac{m_0}{r} - \frac{Q^2}{2r^2}, \tag{3.27}$$

$$\Rightarrow \quad \frac{A_0'}{A_0} = \frac{2}{r}\frac{Q^2 - rm_0}{2rm_0 - 2r^2 - Q^2}, \quad \frac{B_0'}{B_0} = \frac{2}{r}\frac{Q^2 - rm_0}{2rm_0 + 2r^2 - Q^2}. \tag{3.28}$$

In this case adiabatic index Γ would have much dependence on the electric field configuration, pressure and density terms.

4 Summary

The present work is devoted to investigate the consequences of $f(R)$ model and electromagnetic field surrounding the spherically symmetric stars. The model $f(R) = R + \alpha R^n - \frac{\mu^4}{R^m}$ is a viable framework to assume in this dark energy era. The usual matter configuration of star is charged anisotropic fluid which dissipate energy in terms of radiation streaming and heat flow and hence effect stability drastically. As modified field equations are highly complicated non-linear differential equations whose solution has not been established yet. Thus, we have applied perturbation approach, on the dynamical equations to observe evolution of different variables at the surface of a collapsing star.

It is found that instability range of assumed star depend upon the adiabatic index Γ which further depends upon the tangential and radial pressure, density, electric field and radiation etc. It is worthwhile to mention that the electromagnetic effects along with the higher order curvature terms contributions builds up the stability range because dissipative effects decelerate the collapse. The observations for limiting cases $\alpha \to 0, \beta \to 0$ reduce the results to GR.

The strong gravitational field impressions can be anticipated by analyzing the dark source entries appearing in collapse equation (3.19), most of them appears to be negative in equation implying instabilities. We may argue that strong gravitational field limit bring instabilities more rapidly in comparison to the weak field limit implying gravitational collapse. Furthermore, it is worth stressing that the numerical pattern devised in [46]-[48] can be taken into account to address instability problem in an alternate and adequate way. The strong field limit can be established more appropriately by constraining to some specific form of relativistic object and present Jeans analysis.

By comparing our work with the literature, it is observed that by vanishing of parameters involved in the chosen $f(R)$, the obtained reduced expressions for Γ yield consistency with the already work done for the independent $f(R)$ models. The comparison is as follows

- In [37], the effects of Maxwell source and CDTT model on dynamical instability of locally isotropic background were discussed, the results can be retrieved for our consideration by assuming $p_r = p_\perp, m = -1, \mu = \delta, m = 1$ and $\alpha = 0$.

- For $n = 2, \alpha = \delta$ and $q_\alpha = 0$ in our assumptions the findings are in accordance with the work done in [38] for $f(R) = R + \delta R^2$ model on dynamical instability.

- When we take $\mu = 0$, the results support the arguments in [39]. Also, it is obvious that addition of Maxwell invariant describe more general expanding universe with a wider range of instability in $f(R)$ framework.

- The work on generalized CDTT model in the absence of Maxwell source appears to be special case of our model by assuming $E = 0, Q = 0, p_r = p_\perp, n = 2$ and $m = 1$. The results for these substitutions in our considerations are in agreement with the findings of a recent paper [41].

Appendix A: Unperturbed Quantities

$$\begin{aligned}
P_1(r,t) &= \frac{1}{\kappa}\left[A^2\left\{\frac{1}{A^2}\left(\frac{f-Rf_R}{2}-\frac{\dot{f}_R}{A^2}\left(\frac{\dot{B}}{B}+\frac{2\dot{C}}{C}\right)-\frac{f'_R}{B^2}\left(\frac{B'}{B}-\frac{2C'}{C}\right)\right.\right.\right.\\
&\left.\left.+\frac{f''_R}{B^2}\right)\right\}_{,0}+A^2\left\{\frac{1}{A^2B^2}\left(\dot{f}'_R-\frac{A'}{A}\dot{f}_R-\frac{\dot{B}}{B}f'_R\right)\right\}_{,1}-\frac{\dot{f}_R}{A^2}\left\{\left(\frac{3A'}{A}\right.\right.\\
&\left.\left.+\frac{B'}{B}+\frac{2C'}{C}\right)\frac{AA'}{B^2}+\left(\frac{\dot{B}}{B}\right)^2+2\left(\frac{\dot{C}}{C}\right)^2+\frac{3\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{2\dot{C}}{C}\right)\right\}\\
&+\frac{\dot{f}'_R}{B^2}\left(\frac{3A'}{A}+\frac{B'}{B}+\frac{2C'}{C}\right)-\frac{2f'_R}{B^2}\left\{\frac{A'}{A}\left(\frac{2\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\frac{B'}{B}\left(\frac{\dot{A}}{A}\right.\right.\\
&\left.\left.+\frac{\dot{B}}{B}\right)-\frac{C'}{C}\left(\frac{2\dot{A}}{A}-\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right\}+\frac{f''_R}{B^2}\left(\frac{2\dot{A}}{A}+\frac{\dot{B}}{B}\right)+\frac{\dot{A}}{A}(f\\
&\left.-Rf_R)+\frac{\ddot{f}_R}{A^2}\left(\frac{\dot{B}}{B}+\frac{2\dot{C}}{C}\right)\right], \qquad (4.1)\\
P_2(r,t) &= \frac{1}{\kappa}\left[B^2\left\{\frac{1}{B^2}\left(\frac{Rf_R-f}{2}-\frac{\dot{f}_R}{A^2}\left(\frac{\dot{A}}{A}-\frac{2\dot{C}}{C}\right)-\frac{f'_R}{B^2}\left(\frac{A'}{A}+\frac{2C'}{C}\right)\right.\right.\right.\\
&\left.\left.+\frac{\ddot{f}_R}{A^2}\right)\right\}_{,1}+B^2\left\{\frac{1}{A^2B^2}\left(\dot{f_R}'-\frac{A'}{A}\dot{f}_R-\frac{\dot{B}}{B}f'_R\right)\right\}_{,0}+\frac{A'}{A}\left\{\frac{\ddot{f}_R}{A^2}\right.\\
&\left.+\frac{f''_R}{B^2}-\frac{\dot{f}_R}{A^2}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)-\frac{f'_R}{B^2}\left(\frac{A'}{A}+\frac{B'}{B}\right)\right\}+\frac{2B'}{B}\left\{\frac{Rf_R-f}{2}\right.\\
&\left.+\frac{\ddot{f}_R}{A^2}-\frac{\dot{f}_R}{A^2}\left(\frac{\dot{A}}{A}-\frac{2\dot{C}}{C}\right)-\frac{f'_R}{B^2}\left(\frac{A'}{A}+\frac{3C'}{C}\right)\right\}+\frac{1}{A^2}\left(\frac{\dot{A}}{A}+\frac{3\dot{B}}{B}\right.\\
&\left.+\frac{2\dot{C}}{C}\right)\left(\dot{f_R}'-\frac{A'}{A}\dot{f}_R-\frac{\dot{B}}{B}f'_R\right)+\frac{2C'}{C}\left\{\frac{f''_R}{B^2}+\frac{\dot{f}_R}{A^2}\left(\frac{\dot{C}}{C}-\frac{2\dot{B}}{B}\right)\right.\\
&\left.\left.-\frac{f'_R}{B^2}\frac{C'}{C}\right\}\right]. \qquad (4.2)
\end{aligned}$$

Appendix B: Perturbed Quantities

$$
\begin{aligned}
P_{1p} \;=\; & \frac{1}{\kappa}\Bigg[-\Bigg\{\frac{1}{A_0^2B_0^2}\Bigg[\alpha n(n-1)R_0^{n-2}\Bigg(e'+((n-2)R_0^{-1}R_0'-\frac{A_0'}{A_0})e \\
& -\frac{bR_0'}{B_0}\Bigg)+\mu^4 m(m+1)R_0^{-m-2}\Bigg(-e'+(m+2)R_0^{-1}R_0'e+\frac{eA_0'}{A_0} \\
& +\frac{bR_0'}{B_0}\Bigg)\Bigg]\Bigg\}'+\Bigg\{-\frac{1}{A_0^2B_0^2}\Bigg[\alpha(n-1)B_0^2R_0^n\Bigg(\frac{nR_0^{-1}e}{2}+\frac{a}{A_0}\Bigg) \\
& +\mu^4(m+1)B_0^2R_0^{-m}\alpha\Bigg(\frac{-mR_0^{-1}}{2}-\frac{a}{A_0}\Bigg)-n(n-1)R_0^{n-2}\{e'' \\
& +2(n-2)R_0^{-1}R_0'e'-2\Bigg(\frac{a}{A_0}+\frac{b}{B_0}\Bigg)((n-2)R_0^{-1}R_0'^2+R_0'') \\
& +(n-2)(n-3)R_0^{-2}R_0'^2e+(n-2)R_0^{-1}R_0''e-\Bigg[\Bigg(\frac{b}{B_0}\Bigg)'-2\Bigg(\frac{\bar{c}}{r}\Bigg)' \\
& -2\Bigg(\frac{a}{A_0}+\frac{b}{B_0}\Bigg)\Bigg(\frac{B_0'}{B_0}-\frac{2}{r}\Bigg)\Bigg]R_0'-\Bigg(\frac{B_0'}{B_0}-\frac{2}{r}\Bigg)((n-2)R_0^{-1}R_0'e \\
& +e')\}-\mu^4m(m+1)R_0^{-m-2}\big(2(m+2)R_0^{-1}R_0'e'+(m+2)R_0^{-1}R_0''e \\
& -e''-(m+2)(m+3)R_0^{-2}R_0'^2e-2\Bigg(\frac{a}{A_0}+\frac{b}{B_0}\Bigg)((m+2)R_0^{-1}R_0'^2 \\
& -R_0'')-\Bigg[2\Bigg(\frac{a}{A_0}+\frac{b}{B_0}\Bigg)\Bigg(\frac{B_0'}{B_0}-\frac{2}{r}\Bigg)-\Bigg(\frac{b}{B_0}\Bigg)'+2\Bigg(\frac{\bar{c}}{r}\Bigg)'\Bigg]R_0'+((m \\
& +2)R_0^{-1}R_0'e)\Bigg(\frac{B_0'}{B_0}-\frac{2}{r}\Bigg)-e'\Bigg)\Bigg]\Bigg\}_{,0}\alpha(n-1)R_0^n\Bigg\{\frac{2a}{A_0^3B_0^2}\Bigg(-\frac{B_0^2}{2}+ \\
& (n-1)\big[(n-2)R_0^{-1}R_0'^2+R_0''\big]+nR_0^{-2}R_0'\Bigg(\frac{B_0'}{B_0}-\frac{2}{r}\Bigg)\Bigg)+(R_0''+(n \\
& -2)R_0^{-1}R_0'^2-R_0'\Bigg(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\Bigg)\Bigg)\frac{bnR_0^{-2}}{A_0^2B_0^3}+\frac{2n\bar{c}R_0^{-2}R_0'}{rA_0^2B_0^2}\Bigg(\frac{1}{r}-\frac{A_0'}{A_0}\Bigg)- \\
& \frac{nR_0^{-2}}{A_0^2B_0^2}\Bigg(e'+(n-2)R_0^{-1}R_0'e-\frac{eA_0'}{A_0}-\frac{bR_0'}{B_0}\Bigg)+\Bigg(\frac{3A_0'}{A_0}+\frac{B_0'}{B_0}+\frac{2}{r}\Bigg)\Bigg\} \\
& +\mu^4(m+1)R_0^{-m}\Bigg\{\frac{2a}{A_0^3B_0^2}\Bigg(mR_0^{-2}R_0'\Bigg(\frac{B_0'}{B_0}-\frac{2}{r}\Bigg)+\frac{B_0^2}{2}+(m+1)\big[R_0'' \\
& -(m+2)R_0^{-1}R_0'^2\big]\Bigg)+\frac{bmR_0^{-2}}{A_0^2B_0^3}\Bigg((m+2)R_0^{-1}R_0'^2+R_0'\Bigg(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\Bigg) \\
& -R_0''\Bigg)-\frac{mR_0^{-2}}{A_0^2B_0^2}\Bigg(-e'-\frac{2m\bar{c}R_0^{-2}R_0'}{rA_0^2B_0^2}\Bigg(\frac{1}{r}-\frac{A_0'}{A_0}\Bigg)+\frac{eA_0'}{A_0}+\frac{bR_0'}{B_0}
\end{aligned}
$$

$$
\begin{aligned}
&+(m+2)R_0^{-1}R_0'e)\left(\frac{3A_0'}{A_0}+\frac{B_0'}{B_0}+\frac{2}{r}\right)\Big\}\Big], \qquad (4.3)\\
P_{2p} \;=\; & -\frac{1}{\kappa}\left[\frac{\ddot{T}}{T}\left\{\frac{1}{A_0^2B_0^2}\left[\alpha n(n-1)R_0^{n-2}\left(e'+(n-2)R_0^{-1}R_0'e-\frac{eA_0'}{A_0}\right.\right.\right.\right.\\
&\left.-\frac{bR_0'}{B_0}\right)-\mu^4 m(m+1)R_0^{-m-2}\left(e'-(m+2)R_0^{-1}R_0'e-\frac{eA_0'}{A_0}\right.\\
&\left.\left.\left.-\frac{bR_0'}{B_0}\right)\right]\right\}-\left\{\frac{e}{A_0^2B_0^2}\left[\alpha n(n-1)R_0^{n-2}-\mu^4 m(m+1)R_0^{-m-2}\right]\frac{\ddot{T}}{T}\right.\\
&+\alpha(n-1)R_0^{n}\left[\left[\left(\frac{a}{A_0}\right)'+2\left(\frac{\bar{c}}{r}\right)'-\frac{4b}{B_0}\left(\frac{A_0'}{A_0}+\frac{2}{r}\right)\right]R_0'R_0^{-2}\right.\\
&\left.-nR_0^{-1}e-\frac{2b}{B_0}+\left(\frac{A_0'}{A_0}+\frac{2}{r}\right)\left(e'+(n-2)R_0^{-1}R_0'e\right)R_0^{-2}\right]+\mu^4(m\\
&+1)R_0^{-m}\left(mR_0^{-1}e-\left[\left(\frac{a}{A_0}\right)'+2\left(\frac{\bar{c}}{r}\right)'-\frac{4b}{B_0}\left(\frac{A_0'}{A_0}+\frac{2}{r}\right)\right]R_0'R_0^{-2}\right.\\
&\left.\left.-\frac{2b}{B_0}+\left(\frac{A_0'}{A_0}+\frac{2}{r}\right)\left(e'(n-2)R_0^{-1}R_0'e\right)R_0^{-2}\right)\right\}'+\left(\alpha n(n-1)R_0^{n-2}\right.\\
&\left.-\mu^4 m(m+1)R_0^{-m-2}\right)\left(\frac{2eB_0'}{A_0^2B_0^3}-\frac{eA_0'}{A_0^3B_0^2}\right)\frac{\ddot{T}}{T}+\alpha n(n-1)R_0^{n-2}\left(e''\right.\\
&+2(n-2)R_0^{-1}R_0'e'+(n-2)(n-3)R_0^{-2}R_0'^2e+(n-2)R_0^{-1}R_0''e\\
&\left.-\frac{2b}{B_0}\left(R_0''+(n-2)R_0^{-1}R_0'^2\right)\right)\left(-\frac{A_0'}{A_0B_0}-\frac{2}{rB_0^2}\right)+\alpha n(n-1)R_0^{n-2}\\
&\times\left(\left[\left(-\frac{A_0'}{A_0B_0}-\frac{2B_0'}{B_0}\right)\left(\frac{a}{A_0}\right)'+\left(-\frac{A_0'}{A_0B_0}+\frac{2}{rB_0^2}\right)\left(\frac{b}{B_0}\right)'\right.\right.\\
&\left.+\left(\frac{4B_0'}{B_0}+\frac{2}{rB_0^4}\right)\left(\frac{\bar{c}}{r}\right)'+\frac{2A_0'}{A_0B_0^2}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}-\frac{4bB_0'}{r^2}\right)+\frac{8bB_0'}{rB_0^6}\right]R_0'\\
&+(e'+(n-2)R_0^{-1}R_0'e')\left[-\frac{A_0'}{A_0B_0}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}-2B_0'\right)+\frac{2}{rB_0}\left(\frac{B_0'}{B_0^4}\right.\right.\\
&\left.\left.+\frac{1}{rB_0^3}+2B_0'\right)\right]-\alpha n(n-1)R_0^{n-2}\left[\frac{A_0'}{A_0B_0^2}\left((n-2)R_0^{-1}R_0'^2+R_0''\right.\right.\\
&\left.+\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)R_0'\right)\left(\frac{a'}{A_0'}-\frac{a}{A_0}-\frac{2b}{B_0}\right)+\frac{2}{rB_0^4}\left((n-2)R_0^{-1}R_0'^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+R_0''+\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)R_0'\Big)\left(\bar{c}-\frac{1}{r}-\frac{2b}{B_0}\right)\Big]+\alpha(n-1)R_0^n\left[\frac{2b}{B_0}\right.\\
&-nR_0^{-1}e+\frac{1}{B_0^2}\left(\frac{b'}{B_0'}-\frac{b}{B_0}\right)\left(-\frac{1}{2}+\frac{nR_0^{-2}R_0'}{B_0^2}\left(\frac{A_0'}{A_0}+\frac{2}{r}\right)\right)\Big]\frac{2B_0'}{B_0}\\
&+\mu^4m(m+1)R_0^{-m-2}\left(-\frac{A_0'}{A_0B_0}-\frac{2}{rB_0^2}\right)\big(-e''+2(m+2)R_0^{-1}R_0'e\\
&-(m+2)(m+3)R_0^{-2}R_0'^2e+(m+2)R_0^{-1}R_0''e-\frac{2b}{B_0}\big((m+2)R_0^{-1}R_0'^2\\
&-R_0''\big)\big)+\mu^4m(m+1)R_0^{-m-2}\left(-\left[\left(-\frac{A_0'}{A_0B_0}-\frac{2B_0'}{B_0}\right)\left(\frac{a}{A_0}\right)'\right.\right.\\
&+\left(-\frac{A_0'}{A_0B_0}+\frac{2}{rB_0^2}\right)\left(\frac{b}{B_0}\right)'+\left(\frac{4B_0'}{B_0}+\frac{2}{rB_0^4}\right)\left(\frac{\bar{c}}{r}\right)'+\frac{2A_0'}{A_0B_0^2}\left(\frac{A_0'}{A_0}\right.\\
&\left.+\frac{B_0'}{B_0}-\frac{4bB_0'}{r^2}\right)+\frac{8bB_0'}{rB_0^6}\Big]R_0'+(-e'+(m+2)R_0^{-1}R_0'e')\left[\frac{2}{rB_0}\left(\frac{B_0'}{B_0^4}\right.\right.\\
&\left.+\frac{1}{rB_0^3}+2B_0'\right)-\frac{A_0'}{A_0B_0}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}-2B_0'\right)\Big]-\mu^4m(m+1)R_0^{-m-2}\\
&\times\left[\frac{A_0'}{A_0B_0^3}\left(-(m+2)R_0^{-1}R_0'^2+R_0''+\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)R_0'\right)\left(\frac{a'}{A_0'}-\frac{a}{A_0}\right.\right.\\
&\left.-\frac{2b}{B_0}\right)+\frac{2}{rB_0^4}\left(R_0''-(m+2)R_0^{-1}R_0'^2+\left(\frac{B_0'}{B_0}+\frac{1}{r}\right)R_0'\right)+\left(\bar{c}-\frac{1}{r}\right.\\
&\left.\left.-\frac{2b}{B_0}\right)\right]+\mu^4(m+1)R_0^{-m}\left[\frac{1}{B_0^2}\left(\frac{b'}{B_0'}-\frac{b}{B_0}\right)\left(\frac{mR_0^{-2}R_0'}{B_0^2}\left(\frac{A_0'}{A_0}+\frac{2}{r}\right)\right.\right.\\
&\left.+\frac{1}{2}\right)+(mR_0^{-1}e-\frac{2b}{B_0})\Big]\frac{2B_0'}{B_0}\Big]T, \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
P_3 \;=\;& 2\left(\frac{A_0'}{B_0}+\frac{A_0}{rB_0}\right)\left[\frac{\alpha R_0^{n-1}}{\kappa A_0B_0}\big\{(n-1)R_0^{-1}+(n-1)(n-2)R_0^{-2}R_0'e\right.\\
&-(n-1)R_0^{-1}\left(\frac{eA_0'}{A_0}+\frac{b}{B_0}\right)+2\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\Big\}+\frac{\mu^4mR_0^{-m-1}}{\kappa A_0B_0}\\
&\times\ \{-(m+1)R_0^{-1}e'+(m+1)(m+2)R_0^{-2}R_0'e+(m+1)R_0^{-1}\left(\frac{eA_0'}{A_0}\right.\\
&\left.\left.+\frac{b}{B_0}\right)+2\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\right\}+\frac{2}{\kappa A_0B_0}\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\Big]\dot{T}\\
&+\frac{A_0}{B_0}\left[\frac{\alpha R_0^{n-1}}{\kappa A_0B_0}\big\{(n-1)R_0^{-1}+(n-1)(n-2)R_0^{-2}R_0'e-(n-1)R_0^{-1}\right.
\end{aligned}
$$

$$
\begin{aligned}
& \times\ \left(\frac{eA_0'}{A_0}+\frac{b}{B_0}\right)+2\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\Big\}+\frac{\mu^4 mR_0^{-m-1}}{\kappa A_0B_0}\{-R_0^{-1}e'(m \\
& +1)+(m+1)(m+2)R_0^{-2}R_0'e+(m+1)R_0^{-1}\left(\frac{eA_0'}{A_0}+\frac{b}{B_0}\right)+2\left(\frac{\bar{c}'}{r}\right. \\
& \left.-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\Big\}+\frac{2}{\kappa A_0B_0}\left(\frac{\bar{c}'}{r}-\frac{\bar{c}A_0'}{rA_0}-\frac{b}{rB_0}\right)\Big]'\dot{T}+A_0^2P_{1p}, \qquad (4.5)
\end{aligned}
$$

$$
\begin{aligned}
P_4 &= -\frac{rA_0^2B_0}{br+2B_0\bar{c}}\left[\frac{e}{2}-\frac{2\bar{c}}{r^3}-\frac{1}{A_0B_0^2}\left\{A_0''[\frac{a}{A_0}+\frac{2b}{B_0}]-\frac{1}{B_0}\left(a'B_0'+A_0'b'\right.\right.\right. \\
&\left.+a''-A_0'B_0'[\frac{a}{A_0}+\frac{3b}{B_0}]\right)+\frac{2}{r}\left\{a'+\bar{c}'A_0'-A_0'[\frac{a}{A_0}+\frac{2b}{B_0}+\frac{\bar{c}}{r}]\right\} \\
&\left.\left.+\frac{A_0}{r}\left\{\bar{c}''-\frac{b'}{B_0}-\frac{B_0'\bar{c}'}{B_0}+\frac{3b}{B_0}+\frac{\bar{c}}{r}\right\}+\frac{2}{r^2}[\bar{c}'-\frac{b}{B_0}-\frac{\bar{c}}{r}]\right\}\right], \qquad (4.6) \\
P_{2P}^N &= -\frac{1}{\kappa}\left[\left\{\alpha n(n-1)R_0^{n-2}\left(e'+(n-2)R_0^{-1}R_0'e-\mu^4 m(m+1)R_0^{-m-2}\right.\right.\right. \\
&\left.\left.-bR_0'\right)\left(e'-(m+2)R_0^{-1}R_0'e-bR_0'\right)\right\}\frac{\ddot{T}}{T}+\left\{e\left[\alpha n(n-1)R_0^{n-2}\right.\right. \\
&\left.-\mu^4 m(m+1)R_0^{-m-2}\right]\frac{\ddot{T}}{T}+\alpha(n-1)R_0^n\left[-nR_0^{-1}e-2b+\left[2\left(\frac{\bar{c}}{r}\right)'\right.\right. \\
&\left.\left.+a'-\frac{8b}{r}\right]+\frac{2}{r}\left(e'(n-2)R_0^{-1}R_0'e\right)R_0^{-2}\right]R_0'R_0^{-2}+\mu^4(m+1)R_0^{-m} \\
&\times\left(mR_0^{-1}e-2b-\left[a+2\left(\frac{\bar{c}}{r}\right)'-\frac{8b}{r}\right]R_0'R_0^{-2}+\frac{2}{r}\right)\left(R_0^{-2}+e'(n\right. \\
&\left.\left.-2)R_0^{-1}R_0'e\right)\right\}'-\alpha n(n-1)R_0^{n-2}\left(-\frac{2}{r}\right)\left(e''+2(n-2)R_0^{-1}R_0'e'\right. \\
&+(n-2)(n-3)R_0^{-2}R_0'^2e+(n-2)R_0^{-1}R_0''e-2b\left((n-2)R_0^{-1}R_0'^2\right. \\
&\left.\left.+R_0''\right)\right)+\alpha n(n-1)R_0^{n-2}\left(\left[\left(\frac{2}{r}\right)b'+\frac{2}{r}\left(\frac{\bar{c}}{r}\right)'\right]R_0'+\frac{2}{r^2}(e'+(n\right. \\
&\left.-2)R_0^{-1}R_0'e')\right)-\alpha n(n-1)R_0^{n-2}\left[\frac{2}{r}\left(\bar{c}-\frac{1}{r}-2b\right)\left(R_0''+\frac{1}{r}R_0'\right.\right. \\
&\left.\left.+(n-2)R_0^{-1}R_0'^2\right)\right]+\mu^4 m(m+1)R_0^{-m-2}\left(-\frac{2}{r}\right)\left(2(m+2)R_0^{-1}R_0'e\right. \\
&-e''-(m+2)(m+3)R_0^{-2}R_0'^2e-2b\left((m+2)R_0^{-1}R_0'^2-R_0''\right)+(m \\
&\left.+2)R_0^{-1}R_0''e\right)+\mu^4 m(m+1)R_0^{-m-2}\left(-\left[\frac{2b'}{r}+\frac{2}{r}\left(\frac{\bar{c}}{r}\right)'\right]R_0'+(-e'\right.
\end{aligned}
$$

$$
\begin{aligned}
& +(m+2)R_0^{-1}R_0'e'\Big)\left(\frac{2}{r^2}\right) - \mu^4 m(m+1)R_0^{-m-2}\left(R_0'' + \frac{1}{r}R_0'\right. \\
& \left.\left. -(m+2)R_0^{-1}R_0'^2\right)\left(\bar{c} - \frac{1}{r} - 2b\right)\left(\frac{2}{r}\right)\right] T, \qquad (4.7) \\
P_5 \;=\; & P_4\left[\frac{\alpha n R_0^{n-1}}{\kappa}\left[(n-1)R_0^{-1}e' + (n-1)(n-2)R_0^{-2}R_0'e + 2\left(\frac{\bar{c}'}{r}\right.\right.\right. \\
& \left.\left. -\frac{b}{r}\right) - (n-1)R_0'b\right] + \frac{\mu^4 m R_0^{-m-1}}{\kappa}\left[-(m+1)R_0^{-1}e' + (m+1)\right. \\
& \left.\left.(m+2)R_0^{-2}R_0'e + (m+1)R_0^{-1}b\right] + \frac{2}{\kappa}\left(\frac{\bar{c}'}{r} - \frac{b}{r}\right)\right] + P_{2P}^N. \qquad (4.8)
\end{aligned}
$$

References

[1] P. S. Joshi, D. Malafarina, Int. J. Mod. Phys. D **20**, 2641 (2011)

[2] C. Hansen, S. Kawaler, *Stellar Interiors: Physical Principles, Structure and Evolution* (Springer Verlag, 1994)

[3] R. Kippenhahn, A. Weigert, *Stellar Structure and Evolution* (Springer Verlag, 1990)

[4] M. Schwarzschild, *Structure and Evolution of the Stars* (Dover, 1958)

[5] S. Chandrasekhar, Astrophys. J. **140**, 417 (1964)

[6] W. Hillebrandt, K. O. Steinmetz, Astronomy and Astrophysics,**2**, 53 (1976)

[7] R. Chan, et al., MNRAS **239**, 91 (1989)

[8] R. Chan, L. Herrera, N.O. Santos, MNRAS **265**, 533 (1993)

[9] R. Chan, L. Herrera, N.O. Santos, MNRAS **267**, 637 (1994)

[10] R. Chan, et al., MNRAS **316**, 588 (2000)

[11] L. Herrera, N. Santos, MNRAS. **343**, 1207 (2003)

[12] L. Herrera, N.O. Santos, G. Le Denmat, MNRAS **237**, 257 (1989)

[13] L. Herrera, N.O. Santos, G. Le Denmat, Gen. Relativ. Gravit. **44**, 1143 (2012)

[14] L. Herrera, N.O. Santos: Phys. Rev. D **70**, 084004 (2004)

[15] T.P. Sotiriou, V. Faraoni: Rev. Mod. Phys. **82**, 451 (2010)

[16] S Capozziello, V. Faraoni: *Beyond Einstein Gravity* (Springer, New York, 2011)

[17] S. Nojiri, S.D. Odintsov: Phys. Rep. **505**, 59 (2011)

[18] M. Sharif, H.R. Kausar, J. Phys. Soc. Jpn. **80**, 044004 (2011)

[19] M. Sharif, H.R. Kausar, Mod. Phys. Lett. A **25**, 3299 (2010)

[20] M. Sharif, H.R. Kausar, Int. J. Mod. Phys. D **20**, 2239 (2011)

[21] M. Sharif, H.R. Kausar, J. Astrophys. Space Sci. **337**, 805 (2012)

[22] M. Sharif, H.R. Kausar, J. Astrophys. Space Sci. **331**, 281 (2011)

[23] M. Sharif, H.R. Kausar, J. Phys., Conf. Ser. **354**, 012020 (2012)

[24] M. Sharif, H.R. Kausar, JCAP **07**, 022 (2011)

[25] H.R. Kausar, JCAP **01**, 007 (2013)

[26] I. Noureen, M. Zubair, Astrophys. Space Sci. **355**, 2202 (2014)

[27] I. Noureen, M. Zubair, Eur. Phys. J. C **75**, 62 (2015)

[28] M. Zubair, I. Noureen, Eur. Phys. J. C **75**, 265 (2015)

[29] I. Noureen, A.A, Bhatti, M. Zubair, JCAP **02**, 033 (2015)

[30] I. Noureen, M. Zubair, A.A. Bhatti, G. Abbas, Eur. Phys. J. C **75**, 323 (2015)

[31] S.M. Carroll, et al.: New J. Phys. **8**, 323 (2006)

[32] R. Bean, et al.: Phys. Rev. D **75**, 064020 (2007)

[33] F. Schmidt, Phys. Rev. D **78**, 043002 (2008)

[34] Y.S. Song, W. Hu, I. Sawicki, Phys. Rev. D **75**, 044004 (2007)

[35] A.P. Kouretsis, C.G. Tsagas, Phys. Rev. D **82**, 124053 (2010)

[36] M. Sharif, M. Azam, JCAP **2**, 043 (2012)

[37] M. Sharif, Z. Yousaf, MNRAS **434**, 2529 (2013)

[38] M. Sharif, Z. Yousaf, Phys. Rev. D **88**, 024020 (2013)

[39] H.R. Kausar, I. Noureen, Eur. Phys. J. C **74**, 2760 (2014)

[40] M. Sharif, M. Azam, Gen. Relativ. Gravit. **44**, 1181 (2012)

[41] H.R. Kausar, MNRAS **439**, 1536 (2014)

[42] G. Darmois, Memorial des Sciences Mathematiques (Gautheir-Villars, 1927) Fasc. 25.

[43] W. Israel, Nuovo Cimento B **44S10**, 1 (1966)

[44] W. Israel, Erratum **B48**, 463 (1967)

[45] C.W. Misner, D. Sharp, Phys. Rev. **136**, B571 (1964)

[46] S. Capozziello, et al., Phys. Rev. D **85**, 044022 (2012)

[47] S. Capozziello, M. De Laurentis, M. Francaviglia, Astropart. Phys. **29**, 125 (2008)

[48] M. De Laurentis, S. Capozziello, Astropart. Phys. **35**, 257 (2011)