

# Intuitionistic fuzzy similarity measure: Theory and applications

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**Abstract.** First we give notion of integral of intuitionistic fuzzy set and introduce intuitionistic fuzzy implicator and intuitionistic fuzzy inclusion measure. Then we propose a new measure of similarity between two intuitionistic fuzzy sets based on intuitionistic fuzzy inclusion measure. Examples are given to illustrate our notion and the application of this new similarity measure in multi-criteria decision making.

**Keywords:** Intuitionistic fuzzy sets, intuitionistic fuzzy implication, intuitionistic fuzzy inclusion, intuitionistic fuzzy similarity measure, multi-criteria decision making

## 1. Introduction

Similarity is a core element in achieving an understanding of variables that motivate behavior and mediate affect. The concept of similarity measure is used in a variety of scientific fields such as decision making, pattern recognition, machine learning and market prediction. In mathematics, geometric methods for assessing similarity are used in studies of congruence as well as in allied fields such as trigonometry. Topological methods are applied in fields such as semantics. Graph theory is widely used for assessing cladistic similarities in taxonomy. Fuzzy set theory [48] has also developed its own measures of similarity, which find application in areas such as management, medicine and meteorology [4, 17]. The comparisons of different fuzzy similarity measures as well as their aggregations have been studied by Beg and Ashraf [12, 13], Dengfeng and Chuntian [24], Fonck et al. [29], Wang et al. [45], Xu and Xia [47] and Zwick et al. [50]. Cross and Sudkamp [22] gave a notion of similarity by defining fuzzy valued assessment of the similarity of fuzzy sets over a

universe  $X$ . Theoretical and computational properties of the measures was further investigated with the relationships between them [43]. A review, or even a listing of all these similarity measures is impossible. Similarity also plays a key role in the modelling of preference and liking for products or brands, as well as motivations for product consumption. Atanassov [5, 6] introduced the notion of intuitionistic fuzzy set, which is a generalization of fuzzy set and it enriches fuzzy set theory with a notion of indeterminacy expressing hesitation or abstention. Dubois et al. [27] pointed some terminological difficulties in fuzzy set theory and intuitionistic fuzzy set theory in their remarkable paper. Intuitionistic fuzzy set theory has been further extended by several authors, Ban [10] gave the theory of intuitionistic fuzzy measure and its applications. Atanassov [7] in his book gave the detailed survey on the theory and applications of intuitionistic fuzzy set theory and also described new directions and research problems. Measures and integrals for the case of intuitionistic fuzzy sets was formulated in [9] but the concept of similarity measure between any two intuitionistic fuzzy sets based on intuitionistic fuzzy inclusion measure is not introduced so far. Intuitionistic fuzzy sets has been widely applied to decision making problems [15, 19, 23, 37, 39]. One of the important measure amongst intuitionistic fuzzy sets

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is the measure of the similarity between two intuitionistic fuzzy sets, for which several studies have been proposed in the literature. Several of these, however, cannot satisfy the axioms of similarity, and provide counter-intuitive cases [18, 24].

Fuzzy set theory has limitation for the modelling of group decision problems in which vagueness appear in the form of membership and non-membership degrees simultaneously. To overcome this problem, in this paper we propose an algorithm based on intuitionistic fuzzy sets and similarity measure introduced in Section 3. First we propose a method to define an intuitionistic fuzzy relation on a finite universe of discourse by using an intuitionistic triangular norm and an intuitionistic similarity measure between two intuitionistic fuzzy sets. This intuitionistic similarity measure is defined by using an intuitionistic inclusion measure and an intuitionistic fuzzy implication. Our proposed measure takes into account two kinds of distances- one membership and other nonmembership. The remainder of this paper is organized as follows. In Section 2, basic concept of fuzzy sets, t-norm and t-conorm are presented. An overview of intuitionistic fuzzy set, complement, intersection and union of intuitionistic fuzzy sets, intuitionistic fuzzy implications and intuitionistic fuzzy inclusions between intuitionistic fuzzy sets are also discussed. In Section 3, we introduced a notion of integral of intuitionistic fuzzy set, new intuitionistic fuzzy implicator, intuitionistic fuzzy inclusion measure and intuitionistic fuzzy similarity measure between intuitionistic fuzzy sets. In Section 4, we suggest a multi-criteria decision making (MCDM) algorithm of alternatives. Conclusion of paper is given in the last section.

## 2. Basic concepts

A fuzzy set [49]  $A$  in the universe  $X$  is a mapping from  $X$  to  $[0, 1]$  and  $A(x)$  is called the *degree of membership of  $x$  in  $A$*  for any  $x \in X$ . A fuzzy binary relation  $R$  between two crisp sets,  $X$  and  $Y$ , is a fuzzy subset of  $X \times Y$ . In this paper,  $X$  represents a crisp universe of generic elements.

**Definition 2.1.** [42] A binary operation  $T : [0, 1]^2 \rightarrow [0, 1]$  is a triangular norm (t-norm) if it satisfies the following:

1.  $T(1, x) = x$  for all  $x \in X$ . (Boundary condition)
2.  $T(x, y) = T(y, x)$  for all  $x, y \in X$ . (Commutativity)

3.  $T(x, T(y, z)) = T(T(x, y), z)$  for all  $x, y, z \in X$ . (Associativity)
4. If  $w \leq x$  and  $y \leq z$  then  $T(w, y) \leq T(x, z)$  for all  $w, x, y, z \in X$ . (Monotonicity)

To every t-norm  $T$  there corresponds a t-conorm  $S$  called the *dual t-conorm*, defined by:  $S(x, y) = 1 - T(1 - x, 1 - y)$ .

Following are some popular choices for t-norms:

1. The minimum operator  $M : M(x, y) = \min(x, y)$ .
2. The Lukasiewicz operator  $W : W(x, y) = \max(x + y - 1, 0)$ .
3. The product operator  $P : P(x, y) = xy$ .

The corresponding dual t-conorms are:

1. The maximum operator  $M^* : M^*(x, y) = \max(x, y)$ .
2. The bounded sum  $W^* : W^*(x, y) = \min(x + y, 1)$ .
3. The probabilistic sum  $P^* : P^*(x, y) = x + y - xy$ .

**Definition 2.2.** [5] An intuitionistic fuzzy set  $\mathcal{A}$  in  $X$  is:

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)) \mid x \in X\}$$

where  $\mu_{\mathcal{A}}$  and  $\nu_{\mathcal{A}}$  are membership and non-membership functions of  $\mathcal{A}$  respectively. They satisfy  $\mu_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) \leq 1$  for every  $x \in X$ .

If  $\mu_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) = 1$ , for all  $x \in X$ , then the intuitionistic fuzzy set  $\mathcal{A}$  reduces to an ordinary fuzzy set.

An intuitionistic fuzzy set  $\mathcal{A}$  can be defined as a mapping from  $X$  to  $L^* = \{(x_1, x_2) \mid [0, 1]^2 \mid x_1 + x_2 \leq 1\}$  and  $\mathcal{A}(x) = (x_1, x_2)$  is an intuitionistic fuzzy element. The collection of all the intuitionistic fuzzy sets in  $X$  is denoted by  $\mathcal{IF}(X)$ .

Moreover, equipping  $L^*$  with an ordering  $\leq_{L^*}$  defined as  $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$  and  $x_2 \geq y_2$ ,  $(L^*, \leq_{L^*})$  assumes the structure of a complete, bounded lattice with greatest element  $1_{L^*} = (1, 0)$  and smallest element  $0_{L^*} = (0, 1)$ .

Equality  $=_{L^*}$  defined as  $(x_1, x_2) =_{L^*} (y_1, y_2) \Leftrightarrow x_1 = y_1$  and  $x_2 = y_2$ . The sup and inf operations on this lattice are derived from  $\leq_{L^*}$  as:

$$\begin{aligned} \text{sup}((x_1, x_2), (y_1, y_2)) &= (\max(x_1, y_1), \min(x_2, y_2)), \\ \text{inf}((x_1, x_2), (y_1, y_2)) &= (\min(x_1, y_1), \max(x_2, y_2)). \end{aligned}$$

**Definition 2.3.** [6] Let  $\mathcal{A}$  and  $\mathcal{B}$  be two intuitionistic fuzzy sets, following are some important concepts of intuitionistic fuzzy sets:

1. If  $\mathcal{A}(x) \leq_{L^*} \mathcal{B}(x)$  for all  $x \in X$  then  $\mathcal{A} \subseteq \mathcal{B}$ .
2. If  $\mathcal{A}(x) =_{L^*} \mathcal{B}(x)$  for all  $x \in X$  then  $\mathcal{A} = \mathcal{B}$ .
3. The intersection is defined by  $(\mathcal{A} \cap \mathcal{B})(x) = \inf(\mathcal{A}(x), \mathcal{B}(x))$ .
4. The union is defined by  $(\mathcal{A} \cup \mathcal{B})(x) = \sup(\mathcal{A}(x), \mathcal{B}(x))$ .
5. The complement is defined by  $co(\mathcal{A})(x) = (x_2, x_1)$ , for all  $x = (x_1, x_2) \in L^*$ ,

**Definition 2.4.** [25] An intuitionistic t-norm on  $L^*$  is any increasing, commutative, associative mapping  $\mathcal{T}: L^* \times L^* \rightarrow L^*$  that satisfies  $\mathcal{T}(1_{L^*}, x) = x$ , for all  $x \in L^*$ ; an intuitionistic t-conorm on  $L^*$  is any increasing, commutative, associative mapping  $\mathcal{S}: L^* \times L^* \rightarrow L^*$  satisfying  $\mathcal{S}(0_{L^*}, x) = x$ , for all  $x \in L^*$ .

Following mappings  $\mathcal{T}_M$  and  $\mathcal{S}_M$ , are intuitionistic Min t-norm and Max t-conorm:

$$\mathcal{T}_M(x, y) = (M(x_1, y_1), M^*(x_2, y_2)),$$

$$\mathcal{S}_M(x, y) = (M^*(x_1, y_1), M(x_2, y_2)),$$

where  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$  and  $x, y \in L^*$ .

Following mappings  $\mathcal{T}_W$  and  $\mathcal{S}_W$ , are intuitionistic Lukasiewicz t-norm and t-conorm:

$$\mathcal{T}_W(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1)),$$

$$\mathcal{S}_W(x, y) = (\min(1, x_1 + 1 - y_2, y_1 + 1 - x_2), \max(0, x_2 + y_2 - 1)),$$

where  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$  and  $x, y \in L^*$ .

The greatest intuitionistic t-norm with respect to the ordering  $\leq_{L^*}$  is  $\mathcal{T}_M$ , while the smallest intuitionistic t-conorm with respect to  $\leq_{L^*}$  is  $\mathcal{S}_M$ .

Intuitionistic t-norms and intuitionistic t-conorms can be partitioned into two classes by the following:

An intuitionistic t-norm  $\mathcal{T}$  on  $L^*$  (resp. intuitionistic t-conorm  $\mathcal{S}$ ) is called representable if there exists a t-norm  $T$  and its corresponding dual t-conorm  $S$  on  $[0, 1]$  (resp. a t-conorm  $S'$  and its corresponding dual t-norm  $T'$  on  $[0, 1]$ ) such that, for  $x = (x_1, x_2)$ ,  $y = (y_1, y_2) \in L^*$ ,  $\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2))$ ,  $(\mathcal{S}(x, y) = (S'(x_1, y_1), T'(x_2, y_2)))$ ;  $T$  and  $S$  (resp.  $S'$  and  $T'$ ) are called the representants of  $\mathcal{T}$  (resp.  $\mathcal{S}$ ). Clearly, intuitionistic Min t-norm and intuitionistic Max t-conorm are representable.

**Definition 2.5.** [21] An intuitionistic fuzzy implication is a mapping  $\mathcal{I}: L^* \times L^* \rightarrow L^*$  that satisfies, for all  $a, b, c \in L^*$ ,

1.  $a <_{L^*} b \Rightarrow \mathcal{I}(a, c) \geq_{L^*} \mathcal{I}(b, c)$
2.  $b <_{L^*} c \Rightarrow \mathcal{I}(a, b) \leq_{L^*} \mathcal{I}(a, c)$
3.  $\mathcal{I}(0_{L^*}, a) = 1_{L^*}$

4.  $\mathcal{I}(a, 1_{L^*}) = 1_{L^*}$
5.  $\mathcal{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}$

**Definition 2.6.** [6] Let  $\mathcal{A}$  and  $\mathcal{B}$  be two intuitionistic fuzzy sets of  $X$ . The intuitionistic fuzzy set inclusion on  $X$ , is defined as

$$Inc(\mathcal{A}, \mathcal{B}) = \inf_{x \in X} \mathcal{I}(\mathcal{A}(x), \mathcal{B}(x)).$$

where  $\mathcal{I}$  is an intuitionistic fuzzy implicator.

### 3. New concepts; integral, implication, inclusion and similarity measure of intuitionistic fuzzy sets

Up till now there does not exist a valid, general-purpose definition of similarity measure. There do exist several special-purpose definitions which have been employed with success in cluster analysis [31, 33], search [2, 3], classification [11, 30, 35], recognition [28, 36] and diagnostics [1, 34]. In this section, first we give a notion of integral of intuitionistic fuzzy sets and then propose a new form of intuitionistic fuzzy implication, inclusion and give a similarity measure.

**Definition 3.1.** Intuitionistic fuzzy elements  $\mathcal{A}(x) = (a_1, a_2)$  and  $\mathcal{B}(x) = (b_1, b_2)$  are said to be comparable intuitionistic fuzzy elements if these elements satisfy one of the following two conditions.

1.  $a_1 \leq b_1$  and  $b_2 \leq a_2$
2.  $a_1 \geq b_1$  and  $b_2 \geq a_2$

**Remark 3.2.** Intuitionistic fuzzy sets  $\mathcal{A}, \mathcal{B}$  are said to be comparable intuitionistic fuzzy sets if  $\mathcal{A}(x)$  and  $\mathcal{B}(x)$  are comparable intuitionistic fuzzy elements for all  $x \in X$ .

We define intuitionistic product t-norm  $\mathcal{T}_P$  and its respective intuitionistic product t-conorm  $\mathcal{S}_P$  as:

$$\mathcal{T}_P(x, y) = (P(x_1, y_1), P^*(x_2, y_2))$$

$$= (x_1 \cdot y_1, x_2 + y_2 - x_2 \cdot y_2),$$

$$\mathcal{S}_P(x, y) = (P^*(x_1, y_1), P(x_2, y_2))$$

$$= (x_1 + y_1 - x_1 \cdot y_1, x_2 \cdot y_2).$$

**Remark 3.3.** Let  $\tilde{x}_i = (x_i, x'_i)$  ( $i = 1, 2, \dots, n$ ) such that  $\tilde{x}_i \in L^*$ . The associativity of t-norms and t-conorms allows us to extend the product  $P$  and probabilistic sum  $P^*$  in unique way to an  $n$ -array operation in the usual way by induction, defining for each  $n$ -tuple  $(x_1, x_2, \dots, x_n) \in [0, 1]^n$  and  $(x'_1, x'_2, \dots, x'_n) \in [0, 1]^n$ , respectively:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n x_i,$$

$$P^*(x'_1, x'_2, \dots, x'_n) = 1 - \prod_{i=1}^n (1 - x'_i).$$

It further gives us  $T_P$  for  $n$ -tuple  $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \in (L^*)^n$  is:

$$\begin{aligned} T_P(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) &= (P(x_1, x_2, \dots, x_n), P^*(x'_1, x'_2, \dots, x'_n)) \\ &= \left( \prod_{i=1}^n x_i, 1 - \prod_{i=1}^n (1 - x'_i) \right). \end{aligned}$$

**Example 3.4.** Let  $X = [0, 20]$  be the universe of discourse. Define an intuitionistic fuzzy set  $\mathcal{A} \subseteq L^*$  by

$$\mathcal{A}(x) = \begin{cases} (1, 0) & x < 5 \\ (-1/5)x + 2, -1/2 + x/10 & 5 \leq x \leq 10 \\ (0, -1/2 + x/10) & 10 < x < 15 \\ (0, 1) & x \geq 15 \end{cases}$$

The intuitionistic fuzzy set  $\mathcal{A}$  is shown in Fig. 1.

**Example 3.5.** Another intuitionistic fuzzy set  $\mathcal{B} \subseteq L^*$  defined by

$$\mathcal{B}(x) = \begin{cases} (0, 1 - x/10) & x < 5 \\ ((1/5)x - 1, 1 - x/10) & 5 \leq x \leq 10 \\ (-(1/5)x + 3, (x - 10)/5) & 10 < x < 15 \\ (0, 1) & x \geq 15 \end{cases}$$

is shown in Fig. 2.

Intuitionistic fuzzy sets  $\mathcal{A}$  and  $\mathcal{B}$  given in Examples 3.4 and 3.5 are comparable intuitionistic fuzzy sets.

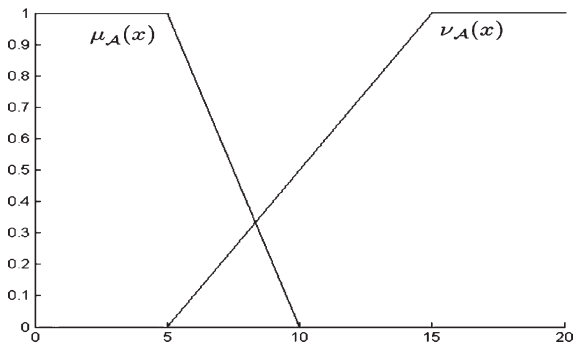


Fig. 1. Intuitionistic fuzzy set  $\mathcal{A}$ .

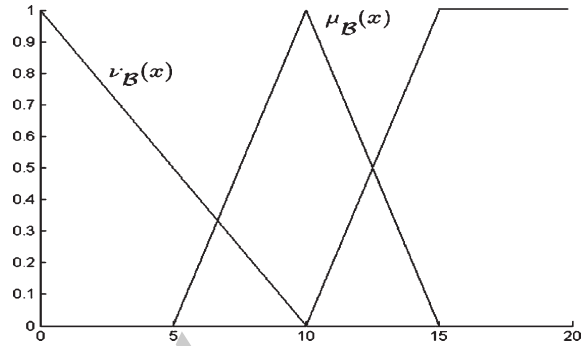


Fig. 2. Intuitionistic fuzzy set  $\mathcal{B}$ .

Membership and non-membership functions of  $\mathcal{A}$  and  $\mathcal{B}$  are bounded and continuous on  $X$ .

**Definition 3.6.** Let  $\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)) \mid x \in X\}$  be an intuitionistic fuzzy set, where  $\mu_{\mathcal{A}}(x)$  and  $\nu_{\mathcal{A}}(x)$  are continuous and bounded functions on  $X$ . Then the integral  $\int_{x \in X} \mathcal{A} dx$  is defined as

$$\begin{aligned} \int_{x \in X} \mathcal{A} dx &= \int_{x \in X} (x, \mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)) dx \\ &= \left( \int_{x \in X} \mu_{\mathcal{A}}(x) dx, \int_{x \in X} \nu_{\mathcal{A}}(x) dx \right). \end{aligned}$$

Bustince and Burillo [20] study the structure of intuitionistic fuzzy relations and characterized it according to the structure of fuzzy relations.

**Definition 3.7.** Intuitionistic fuzzy binary relation  $\mathcal{R}$  from a universe  $X$  to a universe  $Y$  is an intuitionistic fuzzy subset of  $X \times Y$ .

An intuitionistic fuzzy binary relation from  $X$  to  $X$  is called an intuitionistic fuzzy relation on  $X$ .

**Definition 3.8.** An intuitionistic fuzzy relation  $\mathcal{R}$  in a universe  $X$  is called:

1. reflexive if and only if  $\mathcal{R}(x, x) = 1_{L^*}$  for all  $x \in X$ ;
2. symmetric if and only if  $\mathcal{R}(x, y) = \mathcal{R}(y, x)$  for all  $x, y \in X$ ;
3. sup- $\mathcal{T}$  transitive if and only if  $\mathcal{R}(x, z) \geq \sup_{y \in X} \{T(\mathcal{R}(x, y), \mathcal{R}(y, z))\}$  for all  $x, z \in X$ .

Let  $a = (a_1, a_2)$  and  $b = (b_1, b_2)$  be two comparable intuitionistic fuzzy elements then an intuitionistic fuzzy implicator  $\mathcal{I}_{ab}$  is defined as;

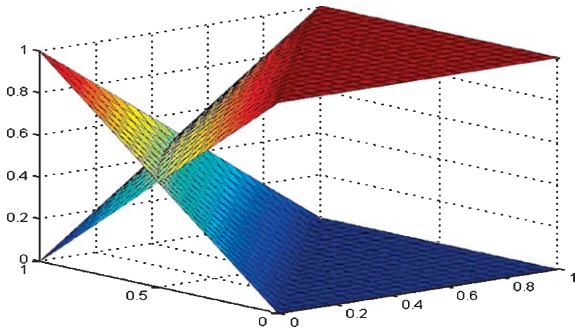


Fig. 3. The implication  $\mathcal{I}_{tab}$ .

$$\mathcal{I}_{tab}(a, b) = \begin{cases} (1, 0) & a_1 \leq b_1, b_2 \leq a_2 \\ (1 - (1 - b_1 + a_1 b_1)(a_1 - b_1), a_1 \geq b_1, b_2 \geq a_2) & (3.1) \\ (1 - a_2 + a_2 b_2)(b_2 - a_2) \end{cases}$$

and its geometry is shown in Fig. 3 below.

**Example 3.9.** Intuitionistic fuzzy implicator for intuitionistic fuzzy set  $\mathcal{A}$  and  $\mathcal{B}$  (from Example 3.4) is given below:

$$\mathcal{I}_{tab}(\mathcal{A}, \mathcal{B}) = \begin{cases} (0, 1 - \frac{x}{10}) & x < 5 \\ (1 - \frac{6x}{5} + \frac{7x^2}{25} - \frac{2x^3}{125}, 5 \leq x < 7.5) \\ (\frac{21}{20} + \frac{17x}{200} - \frac{9x^2}{200} + \frac{x^3}{500}) & \\ (1, 0) & x \geq 7.5 \end{cases}$$

and it is interpreted in Fig. 4.

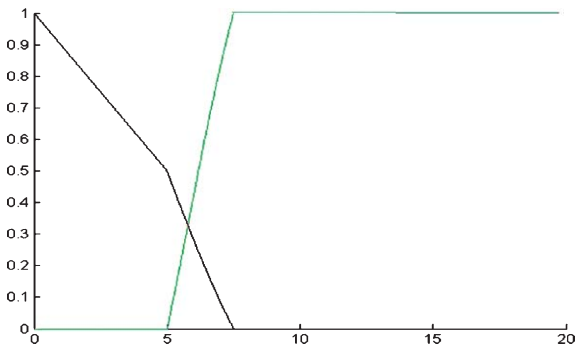


Fig. 4. Intuitionistic implication  $\mathcal{I}_{tab}(\mathcal{A}, \mathcal{B})$ .

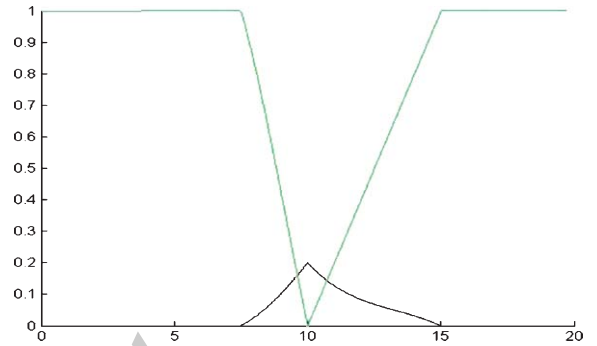


Fig. 5. Intuitionistic implication  $\mathcal{I}_{tab}(\mathcal{B}, \mathcal{A})$ .

Also  $\mathcal{I}_{tab}(\mathcal{B}, \mathcal{A})$  is given below

$$\mathcal{I}_{tab}(\mathcal{B}, \mathcal{A}) = \begin{cases} (1, 0) & x \leq 7.5 \\ (-8 + \frac{18x}{5} - \frac{11x^2}{25} + \frac{2x^3}{125}, & \\ (\frac{3}{4} - \frac{19x}{40} + \frac{13x^2}{200} - \frac{x^3}{500}) & 7.5 < x < 10 \\ (\frac{x}{5} - 2, 6 - \frac{23x}{20} + \frac{8x^2}{100} - \frac{x^3}{500}) & 10 \leq x < 15 \\ (1, 0) & x \geq 15 \end{cases}$$

and it is shown in Fig. 5.

Definition 2.6 is very strong because if there exists an  $x_0 \in X$  such that  $\mathcal{A}(x_0) = (1, 0)$  and  $\mathcal{B}(x_0) = (0, 1)$ , then  $\mathcal{I}nc(\mathcal{A}, \mathcal{B}) = (0, 1)$  even if for most elements of  $X$ ,  $\mathcal{A}(x)$  is very close to the  $\mathcal{B}(x)$ . Therefore, if  $X$  is a closed and bounded subset of set of real numbers  $\mathfrak{R}$ , we can improve this definition of intuitionistic fuzzy set inclusion on  $X$  to overcome this problem.

**Definition 3.10.** Let  $X$  be a closed and bounded subset of  $\mathfrak{R}$  and the universe of discourse,  $\mathcal{A}$  and  $\mathcal{B}$  be the comparable intuitionistic fuzzy sets, where membership and non-membership functions of  $\mathcal{A}$  and  $\mathcal{B}$  are bounded and continuous on  $X$  and  $\mathcal{I}_{tab}$  be the intuitionistic fuzzy implication. The intuitionistic fuzzy set inclusion on  $X$  is defined by

$$\mathcal{I}nc(\mathcal{A}, \mathcal{B}) = \frac{\int_{x \in X} \mathcal{I}_{tab}(\mathcal{A}(x), \mathcal{B}(x)) dx}{\int_{x \in X} dx}. \quad (3.2)$$

**Example 3.11.** Intuitionistic fuzzy set inclusion for intuitionistic fuzzy sets  $\mathcal{A}$  and  $\mathcal{B}$  (from Example 3.4 and 3.5) by using Definition 2.6 is given below:

$$\mathcal{I}nc(\mathcal{A}, \mathcal{B}) = (0.690104, 0.225976),$$

$$\mathcal{I}nc(\mathcal{B}, \mathcal{A}) = (0.815104, 0.086914).$$

Intuitionistic fuzzy set inclusion for intuitionistic fuzzy sets  $\mathcal{A}$  and  $\mathcal{B}$  (from Example 3.4 and 3.5) by using Definition 3.10 is given below:

$$\begin{aligned} \mathcal{I}nc(\mathcal{A}, \mathcal{B}) &= (0, 1), \\ \mathcal{I}nc(\mathcal{B}, \mathcal{A}) &= (0, 0.2). \end{aligned}$$

Above example clearly shows that our newly proposed Definition 3.10 is better than 2 for intuitionistic fuzzy set inclusion.

Next based on our concept of intuitionistic fuzzy set inclusion, we define the intuitionistic similarity measure between intuitionistic fuzzy sets.

**Definition 3.12.** A mapping

$$SM : \mathcal{IF}(X) \times \mathcal{IF}(X) \longrightarrow L^*,$$

is said to be intuitionistic similarity measure between intuitionistic fuzzy sets if  $SM$  satisfies the following:

1.  $SM(\mathcal{A}, \mathcal{B}) = 1_{L^*}$  if  $\mathcal{A} = \mathcal{B}$ ;
2.  $SM(\mathcal{A}, \mathcal{B}) = SM(\mathcal{B}, \mathcal{A})$ ;
3. if  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{IF}(X)$ , such that  $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ , then  $SM(\mathcal{A}, \mathcal{C}) \leq_{L^*} SM(\mathcal{A}, \mathcal{B})$  and  $SM(\mathcal{A}, \mathcal{C}) \leq_{L^*} SM(\mathcal{B}, \mathcal{C})$ .

**Lemma 3.13.** Let a bounded set  $X$  be the universe of discourse,  $\mathcal{A}$  and  $\mathcal{B}$  be two comparable intuitionistic fuzzy sets on  $X$ , and  $T$  be a  $t$ -norm. Then  $SM_0 : \mathcal{IF}(X) \times \mathcal{IF}(X) \longrightarrow L^*$  defined by

$$SM_0(\mathcal{A}, \mathcal{B}) = T(\mathcal{I}nc(\mathcal{A}, \mathcal{B}), \mathcal{I}nc(\mathcal{B}, \mathcal{A})) \quad (3.3)$$

is an intuitionistic similarity measure.

**Proof.** We will show that Equation (3.3) satisfy the axioms of definition 3.12.

1. Let  $SM_0(\mathcal{A}, \mathcal{B}) = 1_{L^*} \iff T(\mathcal{I}nc(\mathcal{A}, \mathcal{B}), \mathcal{I}nc(\mathcal{B}, \mathcal{A})) = 1_{L^*}$  (By Equation (3.3))  
 $\iff \mathcal{I}nc(\mathcal{A}, \mathcal{B}) = 1_{L^*}$  and  $\mathcal{I}nc(\mathcal{B}, \mathcal{A}) = 1_{L^*}$  (By definition of  $T$ )  
 $\iff \mathcal{I}_{tab}(\mathcal{A}(x), \mathcal{B}(x)) = 1_{L^*}$  and  $\mathcal{I}_{tab}(\mathcal{B}(x), \mathcal{A}(x)) = 1_{L^*}$  for all  $x \in X$  (By Equation (3.2))  
 $\iff \mathcal{A}(x) \leq_{L^*} \mathcal{B}(x)$  and  $\mathcal{B}(x) \leq_{L^*} \mathcal{A}(x)$  for all  $x \in X$   
 $\iff \mathcal{A} \subseteq \mathcal{B}$  and  $\mathcal{B} \subseteq \mathcal{A}$   
 $\iff \mathcal{A} = \mathcal{B}$ .
2.  $SM_0(\mathcal{A}, \mathcal{B}) = T(\mathcal{I}nc(\mathcal{A}, \mathcal{B}), \mathcal{I}nc(\mathcal{B}, \mathcal{A}))$  (By Equation (3.3))  
 $SM_0(\mathcal{A}, \mathcal{B}) = T(\mathcal{I}nc(\mathcal{B}, \mathcal{A}), \mathcal{I}nc(\mathcal{A}, \mathcal{B}))$  (As  $T$  is commutative)  
 $SM_0(\mathcal{A}, \mathcal{B}) =_{L^*} SM_0(\mathcal{B}, \mathcal{A})$ .

3. As given  $\mathcal{A} \subseteq \mathcal{B}$   
 $\implies \mathcal{A}(x) \leq_{L^*} \mathcal{B}(x)$  for all  $x \in X$   
 $\implies \mathcal{I}_{tab}(\mathcal{A}(x), \mathcal{B}(x)) = 1_{L^*}$  for all  $x \in X$  (By definition of  $\mathcal{I}_{tab}$ )  
 $\implies \mathcal{I}nc(\mathcal{A}, \mathcal{B}) = 1_{L^*}$  (By Equation (3.2)) By equation (3.3), we know that

$$\begin{aligned} SM_0(\mathcal{A}, \mathcal{B}) &= T(\mathcal{I}nc(\mathcal{A}, \mathcal{B}), \mathcal{I}nc(\mathcal{B}, \mathcal{A})) \\ &= T((1, 0), \mathcal{I}nc(\mathcal{B}, \mathcal{A})) \\ &= \mathcal{I}nc(\mathcal{B}, \mathcal{A}) \\ &= \frac{\int_{x \in X} \mathcal{I}_{tab}(\mathcal{B}(x), \mathcal{A}(x)) dx}{\int_{x \in X} (1, 0) dx}. \end{aligned}$$

- As  $\mathcal{A} \subseteq \mathcal{C}$   
 $\implies \mathcal{A}(x) \leq_{L^*} \mathcal{C}(x)$  for all  $x \in X$   
 $\implies \mathcal{I}_{tab}(\mathcal{A}(x), \mathcal{C}(x)) = 1_{L^*}$  for all  $x \in X$  (By definition of  $\mathcal{I}_{tab}$ )  
 $\implies \mathcal{I}nc(\mathcal{A}, \mathcal{C}) = 1_{L^*}$  (By definition of  $\mathcal{I}nc$ )  
 By Equation (3.3), we know that

$$\begin{aligned} SM_0(\mathcal{A}, \mathcal{C}) &= T(\mathcal{I}nc(\mathcal{A}, \mathcal{C}), \mathcal{I}nc(\mathcal{C}, \mathcal{A})) \\ &= T((1, 0), \mathcal{I}nc(\mathcal{C}, \mathcal{A})) \\ &= \mathcal{I}nc(\mathcal{C}, \mathcal{A}) \\ &= \frac{\int_{x \in X} \mathcal{I}_{tab}(\mathcal{C}(x), \mathcal{A}(x)) dx}{\int_{x \in X} (1, 0) dx}. \end{aligned}$$

By Definition (2.5, 1), we can write that  $\mathcal{I}_{tab}(\mathcal{B}(x), \mathcal{A}(x)) \geq \mathcal{I}_{tab}(\mathcal{C}(x), \mathcal{A}(x))$  for all  $x \in X$ . So we can write that  $\mathcal{I}nc(\mathcal{B}, \mathcal{A}) \geq_{L^*} \mathcal{I}nc(\mathcal{C}, \mathcal{A})$ . It implies that  $SM_0(\mathcal{A}, \mathcal{B}) \geq_{L^*} SM_0(\mathcal{A}, \mathcal{C})$ . Similarly, we can prove that  $SM_0(\mathcal{B}, \mathcal{C}) \geq_{L^*} SM_0(\mathcal{A}, \mathcal{C})$ .  $\square$

The intuitionistic similarity measure between  $\mathcal{A}$  and  $\mathcal{B}$  depends on the degree to which  $\mathcal{A}$  is included and not included in  $\mathcal{B}$ , and the degree to which  $\mathcal{B}$  is included and not included in  $\mathcal{A}$ .

**Example 3.14.** Intuitionistic similarity measure for intuitionistic fuzzy sets  $\mathcal{A}$  and  $\mathcal{B}$  (from Example 3.4 and 3.5) is given below:

$$SM_0(\mathcal{A}, \mathcal{B}) = T_P(Inc(\mathcal{A}, \mathcal{B}), Inc(\mathcal{B}, \mathcal{A})).$$

$$SM_0(\mathcal{A}, \mathcal{B}) = (0.562506, 0.29325).$$

It shows that the degree of similarity and non-similarity between  $\mathcal{A}$  and  $\mathcal{B}$  is 0.562506 and 0.29325, respectively.

**Theorem 3.15.** *Let a bounded set  $X$  be the universe of discourse,  $\mathcal{A}$  and  $\mathcal{B}$  be two intuitionistic fuzzy sets on  $X$ , and  $T$  be a  $t$ -norm. The intuitionistic similarity measure between  $\mathcal{A}$  and  $\mathcal{B}$ , defined by  $SM_0(\mathcal{A}, \mathcal{B}) = T(Inc(\mathcal{A}, \mathcal{B}), Inc(\mathcal{B}, \mathcal{A}))$  satisfies the following:*

1.  $SM_0(\mathcal{A}, co\mathcal{A}) = 1_{L^*}$  if and only if  $\mathcal{A}(x) = (0.5, 0.5)$  for all  $x \in X$ ;
2.  $SM_0(\mathcal{A}, co\mathcal{A}) = 0_{L^*}$  if and only if  $\mathcal{A}(x) = (1, 0)$  for all  $x \in X$  or  $\mathcal{A}(x) = (0, 1)$  for all  $x \in X$ .

**Proof.**

1. Let  $SM_0(\mathcal{A}, co\mathcal{A}) = (1, 0)$   
 $\iff \mathcal{A} = co\mathcal{A}$  (By using lemma 3.13, 1)  
 $\iff \mathcal{A}(x) =_{L^*} co\mathcal{A}(x)$  for all  $x \in X$   
 $\iff \mathcal{A}(x) = (0.5, 0.5)$  for all  $x \in X$ .
2. Let  $SM_0(\mathcal{A}, co\mathcal{A}) = (0, 1)$   
 $\iff T(Inc(\mathcal{A}, co\mathcal{A}), Inc(co\mathcal{A}, \mathcal{A})) = (0, 1)$   
 $\iff Inc(\mathcal{A}, co\mathcal{A}) = (0, 1)$  or  $Inc(co\mathcal{A}, \mathcal{A}) = (0, 1)$ .

Case 1. Consider that  $Inc(\mathcal{A}, co\mathcal{A}) = (0, 1)$

$$\iff \frac{\int_{x \in X} \mathcal{I}_{tab}(\mathcal{A}(x), co\mathcal{A}(x)) dx}{\int_{x \in X} (1, 0) dx} = (0, 1)$$

$$\iff \mathcal{I}_{tab}(\mathcal{A}(x), co\mathcal{A}(x)) = (0, 1) \text{ for all } x \in X$$

$$\iff \mathcal{A}(x) = (1, 0) \text{ for all } x \in X.$$

Case 2. Consider that  $Inc(co\mathcal{A}, \mathcal{A}) = (0, 1)$ .

$$\iff \frac{\int_{x \in X} \mathcal{I}_{tab}(co\mathcal{A}(x), \mathcal{A}(x)) dx}{\int_{x \in X} (1, 0) dx} = (0, 1)$$

$$\iff \mathcal{I}_{tab}(co\mathcal{A}(x), \mathcal{A}(x)) = (0, 1) \text{ for all } x \in X$$

$$\iff co\mathcal{A}(x) = (1, 0) \text{ for all } x \in X.$$

$$\iff \mathcal{A}(x) = (0, 1) \text{ for all } x \in X. \quad \square$$

#### 4. Application to multi-criteria decision making

Real world decision problems are very complex due to the uncertainty in knowledge and modelling. Fuzzy set theory [49] has been successfully applied to handle vagueness and imprecision in an information and it was used to model linguistic terms in decision making problems [26, 44]. The use of linguistic information by experts is quite often in group decision making prob-

lems [14, 16, 32, 40, 41]. But fuzzy set theory has limitation for the modelling of group decision problems in which vagueness appear in the form of membership and non-membership degrees simultaneously. To overcome this problem, here we propose an algorithm based on intuitionistic fuzzy sets and similarity measure introduced in Section 3. Multi-criteria group decision making problem for intuitionistic fuzzy sets was discussed in [8] based on intuitionistic fuzzy relations.

Suppose we have a set of  $n$  criteria and a set of  $m$  alternatives. If there is for each criteria an evaluation on each alternative, then we obtain an  $m \times n$  evaluation matrix. In this matrix, each evaluation is a linguistic term. Our algorithm is based on three main steps: let  $SCE = \{S_1, S_2, \dots, S_n\}$  be the set of scenarios,  $ACT = \{a_1, a_2, \dots, a_m\}$  be the set of actors,  $n, m$  are natural numbers.

Step 1. To obtain an  $m \times n$  evaluation matrix  $E$  whose each element  $E_{ij}$  ( $i \in [1, m], j \in [1, n]$ ) is a linguistic term evaluation for scenario  $S_j$  w.r.t. actor  $a_i$  :

	$S_1$	$S_2$	$\dots$	$S_j$	$\dots$	$S_n$
$a_1$	$E_{11}$	$E_{12}$	$\dots$	$E_{1j}$	$\dots$	$E_{1n}$
$a_2$	$E_{21}$	$E_{22}$	$\dots$	$E_{2j}$	$\dots$	$E_{2n}$
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
$a_i$	$E_{i1}$	$E_{i2}$	$\dots$	$E_{ij}$	$\dots$	$E_{in}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_m$	$E_{m1}$	$E_{m2}$	$\dots$	$E_{mj}$	$\dots$	$E_{mn}$

Each  $E_{ij}$  in this matrix is an intuitionistic fuzzy set on the universe of discourse.

Step 2. To obtain an  $n \times n$  intuitionistic fuzzy relation matrix  $r$  whose each element  $r_{ij} = \mathcal{R}(s_i, s_j)$  ( $i, j \in [1, n]$ ) is the degree to which  $(S_i, S_j)$  is the member of the intuitionistic fuzzy relation  $\mathcal{R}$  :

	$S_1$	$S_2$	$\dots$	$S_j$	$\dots$	$S_n$
$S_1$	$r_{11}$	$r_{12}$	$\dots$	$r_{1j}$	$\dots$	$r_{1n}$
$S_2$	$r_{21}$	$r_{22}$	$\dots$	$r_{2j}$	$\dots$	$r_{2n}$
$\vdots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
$S_i$	$r_{i1}$	$r_{i2}$	$\dots$	$r_{ij}$	$\dots$	$r_{in}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$S_n$	$r_{n1}$	$r_{n2}$	$\dots$	$r_{nj}$	$\dots$	$r_{nn}$

$\mathcal{R}(S_i, S_j)$  is determined based on the similarities between the evaluations for  $S_i$  and  $S_j$  w.r.t. different actors:

$$\mathcal{R}(S_i, S_j) = \mathcal{T}_P \left( \begin{array}{c} \mathcal{SM}_0(E_{1i}, E_{1j}), \\ \mathcal{SM}_0(E_{2i}, E_{2j}), \\ \dots, \mathcal{SM}_0(E_{mi}, E_{mj}) \end{array} \right) \quad (4.1)$$

where  $\mathcal{SM}_0$  is the similarity measure between two comparable intuitionistic fuzzy sets introduced in Equation 3.3.

Step 3. MCDM techniques can be used to partition  $SCE$  based on the  $n \times n$  intuitionistic fuzzy relation matrix  $r$  [38, 46].

Choosing an intuitionistic similarity measure of two fuzzy sets plays an important role in the aforementioned algorithm. According to Equation (3.3) and definition 3.10, this depends on choosing a suitable intuitionistic fuzzy implication.

**Example 4.1.** Let  $Po = \{p_1, p_2\}$  be the set of two policies and  $ACT = \{a_1, a_2, a_3\}$  be the set of three social actors. A matrix  $E$  is given whose each element is an evaluation for a policy on a social actor in view of budget:

$$\begin{array}{ccc} & p_1 & p_2 \\ a_1 & \mathcal{A} & \mathcal{B} \\ a_2 & \mathcal{B} & \mathcal{A} \\ a_3 & \mathcal{A} & \mathcal{A} \end{array},$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are intuitionistic fuzzy sets defined in Examples 3.4 and 3.5 on the universe of discourse  $X = [0, 20]$  with unit ‘million euro’.  $\mathcal{A}$  means that the budget is small and  $\mathcal{B}$  means that the budget is average. Thus according to Equation (4.1), we obtain

$$\begin{aligned} \mathcal{R}(p_1, p_2) &= \mathcal{T}_P(\mathcal{SM}_0(\mathcal{A}, \mathcal{B}), \mathcal{SM}_0(\mathcal{B}, \mathcal{A}), \mathcal{SM}_0(\mathcal{A}, \mathcal{A})) \\ &= \mathcal{T}_P((0.5625, 0.2932), (0.5625, 0.2932), (1, 0)) \\ &= (0.3164, 0.5005). \end{aligned}$$

Thus, it is shown that the degree of relationship and non-relationship between two policies,  $p_1$  and  $p_2$  is 0.3164 and 0.5005, respectively.

## 5. Conclusion

We defined an intuitionistic fuzzy inclusion on the universe of discourse  $\{x\}$  based on an Intuitionistic fuzzy implication  $\mathcal{I}_{tab}$ . There after, we introduced a general intuitionistic inclusion on a universe of discourse  $X$  and intuitionistic similarity measure between two intuitionistic fuzzy sets on  $X$ . Using this a decision

making method has been developed that is capable of dealing with comparative linguistic expressions based on intuitionistic fuzzy sets. Examples have been given to illustrate the concept of intuitionistic fuzzy sets, intuitionistic similarity between two intuitionistic fuzzy sets, and intuitionistic fuzzy relation based on intuitionistic similarity between two intuitionistic fuzzy sets. An algorithm is described to make a MCDM method. In Step 2 of this algorithm, we have to obtain an intuitionistic fuzzy relation on the universe of discourse which is defined by the intuitionistic product t-norm and the intuitionistic similarity between two intuitionistic fuzzy sets. It is shown that to choose a suitable intuitionistic similarity between two intuitionistic fuzzy sets in this step depends on choosing a suitable intuitionistic fuzzy implication because of Lemma 3.13 and Definition 3.10. A simple example is constructed to show the usefulness of the method proposed.

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