Factorization properties and chain conditions on ideals: A linkage

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Abstract: The purpose of this study is to find relationship among the various domains. In particular, the domains possessing factorization properties and the domains which hold different chain conditions on ideals.
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Introduction

Techniques used in literature to explore different properties in rings and domains include factorization properties among elements (e.g., *UFD, FFD, HFD, BFD* etc.), decomposition of ideals (e.g., Noetherian domains, Laskerian domains) and chain conditions on ideals.

Rings and domains satisfying chain condition on ideals form important class of rings and domains in commutative algebra. Initially there are two classes of such rings. Noetherian rings satisfy *ACC* (ascending chain condition) for all ideals and Artinian rings which satisfy *DCC* (descending chain condition). It is known that Artinian rings are Noetherian. Some of other domains satisfying chain condition on ideals include, Mori domains (*ACC* on divisorial ideals), Laskerian domains (*ACC* on prime ideals), strongly Laskerian domains (*ACC* on prime and principal ideals), strongly Hopfian domains (*ACC* on ann $(a) \subseteq$ ann $(a^2) \subseteq$) and domains satisfying *ACC* on principal ideals. In the implication tables C_1, C_2, C_3, C_4 and C_5 are representing the connectors.

Discovering the links

The motivating factor behind this study is the work of D. D. Anderson, D. F. Anderson and M. Zafrullah [1], Kabbaj and Mimouni [18, Figure 1] and Bourbaki [5]. We combine the work of [1, 5, and 18] and add some more implications.

Recall that a ring *R* is Noetherian if every ideal of *R* is finitely generated or any ascending chain of ideals in *R* is stationary, i.e. *R* has *ACCI* (ascending chain condition on ideals). Emmy Noether has proved that every ideal in a

Noetherian ring is a finite intersection of primary ideals. In the terminology of Bourbaki [5, Ch 4, Page 295 and 298] a ring *R* is Laskerian if each ideal of *R* can be written as a finite intersection of primary ideals, and *R* is strongly Laskerian if each ideal of *R* can be written as a finite intersection of strongly primary ideals. It is well known that

Noetherian $\Rightarrow$ strongly Laskerian $\Rightarrow$ Laskerian

In [21] Emmy Noether has proved that every Noetherian ring is Laskerian. But the converse is not true. We give a counter example from [9].

Let *K* be a field and *R* be the set equivalence classes of elements of the form $f(X, Y)/g(X, Y)$ where *X, Y* are indeterminate over *K*, $f, g\in K[X, Y]$, *X* does not divide *g* (in $K[X, Y]$) and $f(0,Y)/g(0,Y)\in K$. Make *R* into a ring by the usual addition and multiplication of rational functions. Then, it can be shown that *R* is a commutative ring that is not Noetherian and every ideal of *R* has a primary decomposition. Also a non discrete *1*-dimensional valuation ring is Laskerian but not strongly Laskerian. Note that this ring does not have *ACC* on principal ideals.

The ring $R=$□$+X$□$[X]$ is an example of non- Noetherian strongly Laskerian domain.

Recall that an integral domain has *ACC* on principal ideals if every ascending chain of principal ideals stabilizes. It is well known that if an integral domain has *ACC* on principal ideals then it is atomic. Following example from [12] shows that an atomic domain does not imply *ACCP*.

Let *F* be a field and *T* additive submonoid of □$^+$ generated by {1/3, 1/(2.5), …,$1/(2^j p_j)$,…}, where $p_0=3$ and $p_1=5$ is the sequence of odd primes. Let *R* be monoid domain *F[X; T]* and $N=\{f\in R$ such that *f* has nonzero constant term}. Then $A=F[X; T]_N$ is an atomic domain which does not satisfy *ACC* on principal ideals.

Following [25], an integral domain *D* is a Mori domain if it satisfies *ACC* on *v*-ideals. Obviously Noetherian domains are Mori domains. Mori domains have the ascending chain condition on principal ideals; they are Archimedean (cf. [3, p. 353]).
Following [7] a ring *R* is a zero divisor ring, *Z D*- ring, if $Z_R(R/I)$ is a finite union of prime ideals for all ideals *I* of *R*. Further in [7, Proposition 7] Evans has shown that Laskerian rings are *Z D*-rings. The converse is false, for Bourbaki [1, p. 170, Exercise 19] shows that a rank *2* valuation ring is not Laskerian. Any valuation ring is *Z D*. since in it any union of primes is a prime. Following [17] a commutative ring R is said to be strongly Hopfian if the chain of annihilators $\text{ann}(a)\subseteq \text{ann}(a^2)\subseteq \dots$ stabilizes for each $a\in R$ and Laskerian are strongly Hopfian.
A ring *R* is called an N-ring if for each ideal *I* of *R* there exists a Noetherian ring *T* containing *R* as a unitary subring with $IT\cap R=I$ [11, 14, 15],· N-rings are Laskerian [11, Proposition 2.14]. By [2] *R* is said to be a *Q* -ring, if every ideal is a finite product of primary ideals. By [2, Theorem 13] *R* is a *Q* -ring if and only if *R* is a Laskerian ring in which every non-maximal prime ideal is quasi-principal. The ring *R* is I_1 if, given any proper ideal *H* and any prime *P*, S_{PH} [illegible] $H : f$ for some f not in *P*. The saturations of *H* through $R\backslash P$ is the same as the conductor ideal that drives *f* into *H* and a ring *R* is I_2 if, given any descending chain of multiplicatively closed sets, and an ideal *J*, the saturations of *J* become constant. I_1 And I_2 are generalizations of Laskerian rings. By [10, Page 456, Exercise 7] a valuation ring is Laskerian (respectively strongly Laskerian) if and only if it is *1*-dimensional (respectively *DVR* of dimension 1). Following [10, Page 79] an integral domain *D* is a Bezout if every finitely generated ideal of *D* is principal. By [10, Theorem 17.1] every valuation ring is a Bezout domain. It is well known that a Bezout domain is a *GCD* domain but a *GCD* domain is Bezout if and only if it is a Prüfer domain. Any Bezout domain is *QR*, in the sense that each of its overrings is localization. Indeed, a Bezout domain may be characterized as a *GCD* domain which is *QR*. An integral domain *D* is said to be a Prüfer domain if D_M is a valuation ring for each maximal ideal *M* of *D* and a domain *D* with the property that each overring of *D* is a quotient ring of *D* is said to have the *QR* -property. A domain

which has the *QR*-property is necessarily Prüfer. (A Prüfer domain need not be a Bezout domain. **Z** [√(-5)] is a Noetherian Prüfer domain that is not a Bezout domain (see [10, Page 278])). Pendleton in [22, Page 500] has shown that a Prüfer domain *D* has the *QR*- property if and only if the radical of each finitely generated ideal of *D* is the radical of a principal ideal. Recall that a domain *D* is a *QQR*-domain if each overring of *D* is an intersection of localizations at prime ideals of *D*. Davis [6] showed that a Prüfer domain must have the *QQR* -property.

An integral domain *D* is atomic if every nonzero nonunit of *D* can be factored as a product of irreducible elements of *D*. Following [1] an integral domain *D* is strongly atomic if for each a, b∈ D^o (non zero elements of *D*), we can write $a = a_1 ... a_s c$ and $b = a_1 ... a_s d$ where $a_1, ..., a_s \in D$ (s ¤ 0) are irreducible and *c, d* ∈ *D* satisfy *gcd(c, d) =1*.

A commutative integral domain *D* (with or without unity) is called a Euclidean domain (*ED*) if there is a map *d;* D^o →Z^+ (where D^o is set of nonzero element of *D*) such that

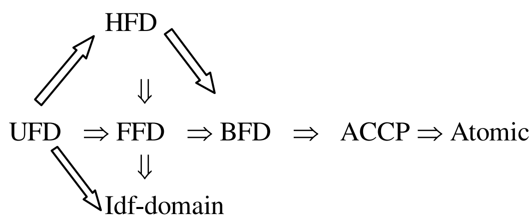
(i) ∀ a, b5 D^o, a | b implies that d(a)″ d(b) or equivalently, d(x) ″ d(xy) for all x, y 5 D^o.

(ii) Given a 5 *D*, b5 D^o there exist *q, r 5 D* such that *a= bq+r* with either *r=0* or else *d(r) < d(b)*.

Recall that a principal ideal domain (*PID*) is an integral domain such that every ideal can be generated by a single element (i. e. every ideal is a principal ideal). By [19, Theorem 4.3.1] every *ED* is a *PID*, but the converse is not true because ring *R=* **Z** *[(1+i√19)/2]* or the ring of even integers *R=2* **Z** are *PID' s* which are not *ED*. All *PIDs* are *UFDs* the converse is true only if every nontrivial prime ideal is maximal.

In the terminology of Kaplansky [20] a *GCD*- domain is a commutative integral domain in which each pair of elements has a greatest common divisor. A *GCD*- domain is a generalization of unique factorization domain (*UFD*).An integral domain *D* is a weak *GCD* domain if for each a, b∈ D^o (nonzero elements of *D*) there are c, $a', b' \in D$ so that $a = ca'$ and $b = cb'$, where gcd(a', b')=*1*.

In [23, Corollary 1] authors have proved that a strongly Laskerian domain is a weak *GCD* domain. The terminology of half-factorial domain (*HFD*) was introduced by Zaks in [26 and27] as a generalization of unique factorization domain (*UFD*) defined as an atomic integral domain *D* is a half factorial domain (*HFD*) if for any irreducible elements $a_1,...,a_n$ and $b_1,...b_m$ of *D* with $a_1 ... a_n = b_1 ... b_m$, then *m = n*. Clearly a *UFD* (unique factorization domain) is also an *HFD*. Call an integral domain *D* a finite factorization domain (*FFD*) if every nonzero nonunit element of *D* is either irreducible or a product of irreducible. *FFD* is a much weaker concept than *UFD*. In [13] Grams and Warner introduced idf –domain (for irreducible-divisor- finite) as: a domain *D* is *idf*-domain if every nonzero element of *D* has at most a finite number of non associate irreducible divisors. An *idf* -domain need not to be atomic. The *idf*-property does not imply any other factorization property. Following [1] a domain *D* is a bounded factorization domain (*BFD*) if *D* is atomic and for each nonzero nonunit of *D* there is a bound on length of factorization into product of irreducible elements. The following implications are clear in [1, Page 2].

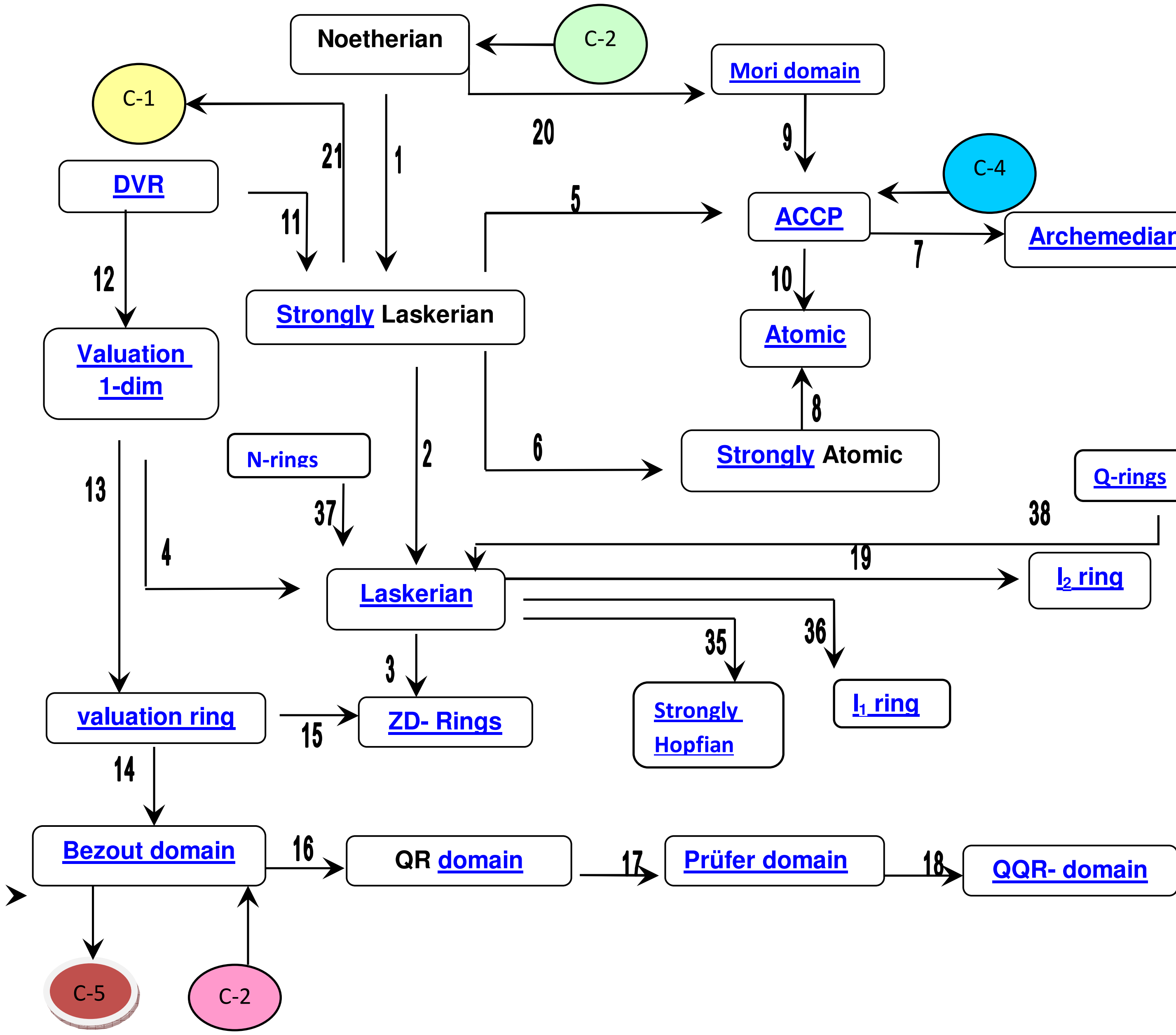


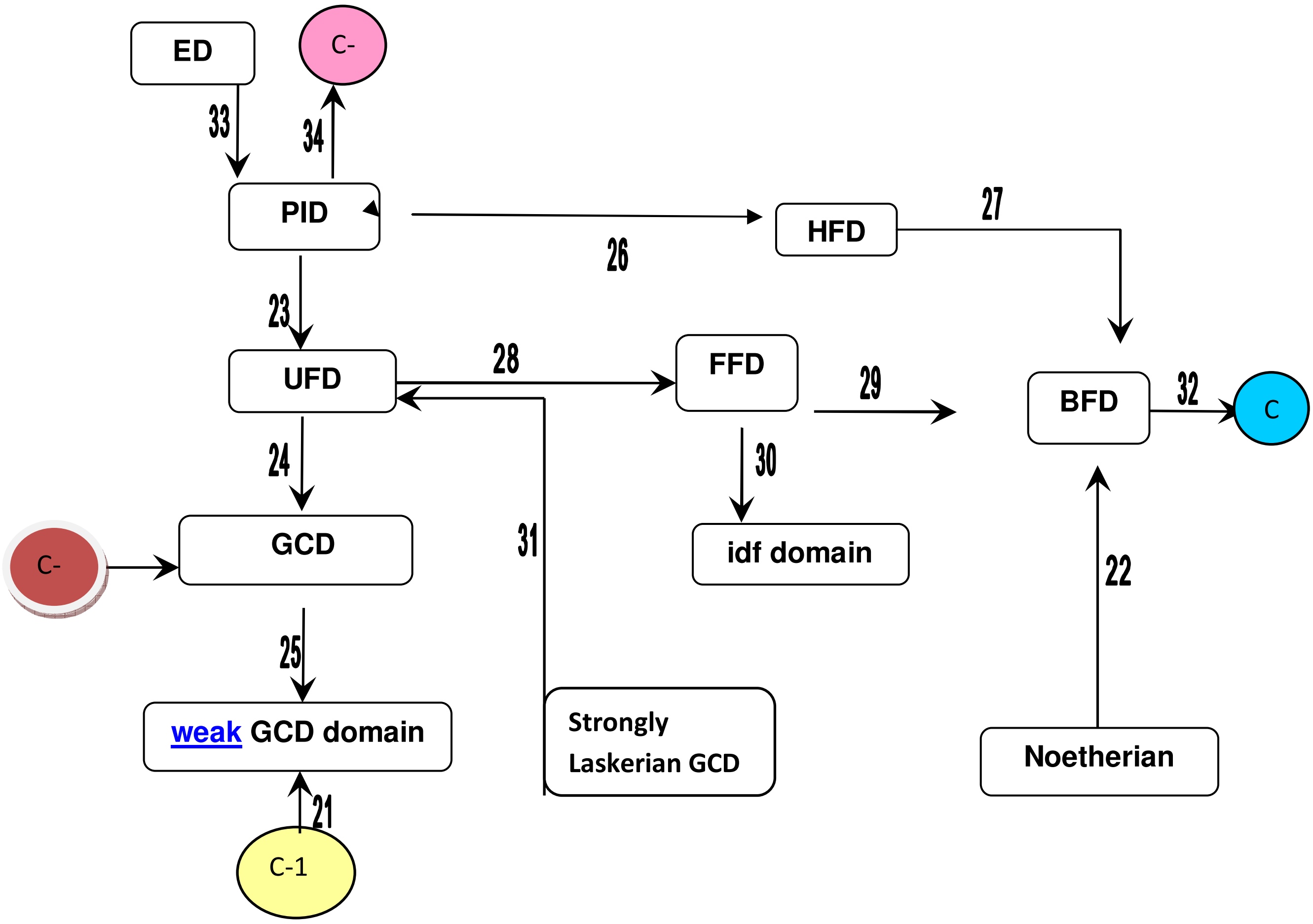
By [1, Proposition 2.2] a Noetherian domain is *BFD*. But converse is not true. For any pair of fields $K_1 \subseteq K_2$, $K_1 + XK_2[X]$ and $K_1 + XK_2[[X]]$ are *BFD's*. Note that they are not Noetherian if $[K_2:K_1]$ is infinite (see [1, Page 10]). However *BFD's* satisfy *ACC* on principal ideals but the converse is not true due to [1, Example 2.1].

In [23, Proposition 1] Shah and Saeed have proved that strongly Laskerian domains satisfy *ACCP* and are strongly atomic. Following [24], an integral domain *D* is Archimedean in case $\bigcap_{n \geq 1} Dr^n = 0$ for each non unit $r \in D$. The most natural examples of Archimedean domains are arbitrary completely integrally closed domains, arbitrary one -dimensional integral domains and arbitrary Noetherian integral domains. By [4, Theorem 2.1] or by [16, Proposition 2.2] if a ring satisfy *ACCP* then it is Archimedean.

Now in the following tables we organize the whole above discussion.

TABLE OF IMPLICATIONS




ED
C-
33
34
PID
26
HFD
27
23
UFD
28
FFD
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BFD
32
C
24
30
C-
GCD
31
idf domain
22
25
weak GCD domain
Strongly
Laskerian GCD
Noetherian
21
C-1

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