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Suzuki type fixed point theorems for generalized multi-valued mappings on a set endowed with two *b*-metrics

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ABSTRACT

In this paper, we obtained Suzuki type fixed point results for a generalized multi-valued mapping on a set equipped with two *b*-metrics. As a consequence, existence and uniqueness of solution of functional equation arising in dynamical programming is also derived.

1. Introduction and preliminaries

Let (X,d) be a metric space and $CB(X)$ be a family of all nonempty closed and bounded subsets of X. A Hausdorff metric H induced by the metric d of X is given by

$$H(A,B)=\max\left\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(y,A)\right\}$$

for every $A,B\in CB(X)$. A multi-valued mapping $T:X\to CB(X)$ is said to be a *contraction* if there exists a constant $k\in[0,1)$ such that for any $x,y\in X$,

$$H(Tx,Ty)\leqslant kd(x,y).$$

A point $x\in X$ is called a fixed point of T if $x\in Tx$. In 1969 Nadler [19] obtained the following multi-valued version of Banach contraction principle.

Theorem 1.1. *Let (X,d) be a complete metric space and $T:X\to CB(X)$ be a contraction. Then T has a fixed point.*

Nadler's multi-valued contraction theorem [19] (see also [9]) was subsequently generalized among others by [6–8,21,17,11,22,15,16]. Recently Djorić and Lazović [12] proved the following Suzuki type fixed point theorem.

Theorem 1.2 [12]. *Let (X,d) is a complete metric space and $T:X\to CB(X)$. If there exist $0\leqslant r<1$ and a non-increasing function φ from $[0,1)$ into $(0,1]$ defined by*

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$$\varphi(r)=\begin{cases}1 & 0\leqslant r<\frac{1}{2}\\ 1-r & \frac{1}{2}\leqslant r<1\end{cases} \tag{1.1}$$

and

$$\varphi(r)d(x,Tx)\leqslant d(x,y) \text{ implies}$$
$$H(Tx,Ty)\leqslant r\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

for all $x,y\in X$, then T has a fixed point.

The concept of b-metric space was introduced by Czerwik in [10]. Several fixed point results for single and multi-valued mappings have been established in the framework of b-metric spaces (see [4,10,24]). In this paper we obtain Kikkawa-Suzuki type fixed point theorems for multi-valued mappings by using two b-metrics. Our results extend and unify recently obtained results for multi-valued mappings. Moreover, using these results we have shown the existence of solution of functional equation arising in dynamic programming.

Definition 1.3 [10]. Let X be any nonempty set and $s\geqslant 1$ be a given real number. A mapping $d:X\times X\to R_+$ is said to be a b-metric if and only if for all $x,y,z\in X$, the following conditions are satisfied:
(d_1) $d(x,y)=0$ if and only if $x=y$;
(d_2) $d(x,y)=d(y,x)$;
(d_3) $d(x,z)\leqslant s[d(x,y)+d(y,z)]$.
A pair (X,d) is called a b-metric space. Following are some examples of b-metric spaces.

Example 1.4 [4]. The space $l_p=\{(x_n)_{n\in\mathbb{N}}\subset R\}$, where $\sum_{n=1}^{\infty}x_n<\infty\},(0<p<1)$, together with the mapping $d:l_p\times l_p\to R$ defined by

$$d(x,y)=\left(\sum_{n=1}^{\infty}|x_n-y_n|^p\right)^{1/p}$$

is a b-metric. Note that

$$d(x,z)\leqslant 2^{1/p}[d(x,y)+d(y,z)]$$

with $s=2^{1/p}>1$.

Example 1.5 [4]. The space $L_p(0<p<1)$ of all real functions $x(t),t\in[0,1]$ such that $\int_0^1|x(t)|^p dt<\infty$ is a b-metric space equipped by b-metric given by

$$d(x,y)=\left(\int_0^1|x(t)-y(t)|^p dt\right)^{1/p}.$$

The constant s is same as obtained in the previous example.

Definition 1.6. Let (X,d) be a b-metric space. A sequence $\{x_n\}_{n\in N}$ in X is called:

(a) Cauchy if and only if for $\epsilon>0$, there exists $n(\epsilon)\in N$ such that for each $n,m\geqslant n(\epsilon)$ we have $d(x_n,x_m)<\epsilon$.
(b) Convergent if and only if there exists $x\in X$ such that for all $\epsilon>0$ there exists $n(\epsilon)\in N$ such that for all $n\geqslant n(\epsilon)$ we have $d(x_n,x)<\epsilon$. In this case we write $\lim_{n\to\infty}x_n=x$.

It is known that a sequence $\{x_n\}_{n\in N}$ in b-metric space X is Cauchy if and only if $\lim_{n\to\infty}d(x_n,x_{n+p})=0$ for all $p\in N$. A sequence $\{x_n\}_{n\in N}$ is convergent to $x\in X$ if and only if $\lim_{n\to\infty}d(x_n,x)=0$. A subset $Y\subset X$ is called closed if and only if for each sequence $\{x_n\}_{n\in N}$ in Y which converges to an element x, we have $x\in Y$. A subset $Y\subset X$ is called bounded if $diam(Y)$ is finite, where

$$diam:P(X)\to R_+\cup\{\infty\},\ \ diam(Y)=\sup\{d(a,b),a,b\in Y\}$$

is the generalized diameter functional, where $P(X)$ is the collection of all nonempty subsets of a b-metric space. A b-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent in X. In the sequel, we denote $CB(X)$ by set of all closed and bounded subsets of a b-metric space (X,d). In consistent with Monica [18], let $D:P(X)\times P(X)\to R_+\cup\{\infty\}$ be defined by

$$D(A,B)=\inf\{d(a,b)|a\in A,b\in B\}$$

for any $A, B \subset X$, here D is called the gap functional between A and B. In particular if $x_0 \in X$, then

$$D(x_0, B) := D(\{x_0\}, B).$$

Let $H : P(X) \times P(X) \to R_+ \cup \{\infty\}$ be defined by

$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(A,y)\right\}$$

for any $A, B \subset X$. Here H is called (generalized) Pompeiu–Hausdorff functional between A and B. We state the following known results from [10] needed in the sequel.

Lemma 1.7. *Let (X, d) be a b-metric space. We have following:*

(i) *For $A \in P(X)$ and $x \in X$,*

$$D(x, A) \leqslant s[d(x,y) + D(y,A)], \text{ for all } x, y \in X.$$

(ii) *For $\{x_n\}_{k=0}^{n} \subset X$,*

$$d(x_n, x_0) \leqslant sd(x_0, x_1) + \cdots + s^{n-1}d(x_{n-2}, x_{n-1}) + s^{n-1}d(x_{n-1}, x_n).$$

(iii) *For $A, B, C \in P(X)$ we have*

$$H(A, C) \leqslant s[H(A,B) + H(B,C)].$$

(iv) *For $A, B \in CL(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that*

$$d(a,b) \leqslant H(A,B) + \alpha,$$

$A, B \in CB(X)$. Also for all $b \in B$ there exists $a \in A$ such that

$$d(a,b) \leqslant sH(A,B).$$

Remark. Let (X, d) be a b-metric space. Then $D(A,B)$ and $H(A,B)$ will be denoted throughout in this work by $D_d(A,B)$ and $H_d\{d\}(A,B)$, respectively with respect to b-metric space (X, d).

2. Fixed point theorems

Now we shall prove our main fixed point theorem.

Theorem 2.1. *Let X be a nonempty set, d and ρ be two b-metrics on X with constants $s \geqslant 1$ and $t \geqslant 1$, respectively and $T : X \to CB(X)$. Let $\varphi : [0,1) \to (0,1]$ be defined as in Theorem 1.2, that is,*

$$\varphi(r) = \begin{cases} 1 & 0 \leqslant r < \frac{1}{2} \\ 1 - r & \frac{1}{2} \leqslant r < 1 \end{cases} \tag{2}$$

Suppose that (X, d) is a complete b-metric space and there exists $c > 0$ such that $d(x,y) \leqslant c\rho(x,y)$ for all $x, y \in X$. If there exist $0 \leqslant r < \min\{\frac{1}{t}, \frac{1}{s}\}$ such that for all $x, y \in X$

$$\frac{\varphi(r)}{t} D_\rho(x, Tx) \leqslant \rho(x,y) \text{ implies } H_\rho(Tx, Ty) \leqslant \frac{r}{t} M_\rho^T(x,y) \tag{2.1}$$

and

$$\frac{\varphi(r)}{s} D_d(x, Tx) \leqslant d(x,y) \text{ implies } H_d(Tx, Ty) \leqslant \frac{r}{s} M_d^T(x,y), \tag{2.2}$$

where

$$M_\rho^T(x,y) = \max\left\{\rho(x,y), D_\rho(x,Tx), D_\rho(y,Ty), \frac{D_\rho(x,Ty) + D_\rho(y,Tx)}{2}\right\}$$

and

$$M_d^T(x,y) = \max\left\{d(x,y), D_d(x,Tx), D_d(y,Ty), \frac{D_d(x,Ty) + D_d(y,Tx)}{2}\right\}$$

Then T has a fixed point.

Proof. Suppose that r_1 and q are real numbers such that $0 \leqslant r < r_1 < 1$ and $1 < q < \frac{1}{tr_1}$. Let $u_1 \in X$ and $u_2 \in Tu_1$. Then there exists $u_3 \in Tu_2$ such that

$$\rho(u_2,u_3) \leqslant qH_\rho(Tu_1,Tu_2).$$

Since $\frac{\varphi(r)}{t} < 1$, then we have

$$\frac{\varphi(r)}{t}D_\rho(u_1,Tu_1) \leqslant D_\rho(u_1,Tu_1) \leqslant \rho(u_1,u_2).$$

Now from (2.1) we have

$$\begin{aligned}\rho(u_2,u_3) \leqslant qH_\rho(Tu_1,Tu_2) &\leqslant \frac{rq}{t}\max\left\{\rho(u_1,u_2),D_\rho(u_1,Tu_1),D_\rho(u_2,Tu_2),\frac{D_\rho(u_1,Tu_2)+D_\rho(u_2,Tu_1)}{2}\right\}\\ &\leqslant \frac{r_1q}{t}\max\left\{\rho(u_1,u_2),\rho(u_2,u_3),\frac{1}{2}\rho(u_1,u_3)\right\}.\end{aligned}$$

Thus

$$\rho(u_2,u_3) \leqslant \frac{r_1q}{t}\max\left\{\rho(u_1,u_2),\rho(u_2,u_3),\frac{1}{2}\rho(u_1,u_3)\right\}. \tag{2.2a}$$

If $\max\{\rho(u_1,u_2),\rho(u_2,u_3),\frac{1}{2}\rho(u_1,u_3)\} = \rho(u_1,u_2)$, then from (2.2a),

$$\rho(u_2,u_3) \leqslant \frac{r_1q}{t}\rho(u_1,u_2).$$

If $\max\{\rho(u_1,u_2),\rho(u_2,u_3),\frac{1}{2}\rho(u_1,u_3)\} = \rho(u_2,u_3)$, then

$$\rho(u_2,u_3) \leqslant \frac{r_1q}{t}\rho(u_2,u_3)$$

implies that $\rho(u_2,u_3) = 0$ which further implies that $u_3 = u_2 \in Tu_2$.

If $\max\{\rho(u_1,u_2),\rho(u_2,u_3),\frac{1}{2}\rho(u_1,u_3)\} = \frac{1}{2}\rho(u_1,u_3)$, then from (2.2a) we have

$$\rho(u_2,u_3) \leqslant \frac{r_1q}{2t}\rho(u_1,u_3) \leqslant \frac{r_1qt}{2t}\rho(u_1,u_2) + \frac{r_1qt}{2t}\rho(u_2,u_3)$$

and hence

$$\rho(u_2,u_3) \leqslant \frac{r_1q}{2-r_1q}\rho(u_1,u_2).$$

Therefore,

$$\rho(u_2,u_3) \leqslant \max\left\{\frac{r_1q}{t},\frac{r_1q}{2-r_1q}\right\}\rho(u_1,u_2).$$

Continuing this process, we obtain a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$\rho(u_n,u_{n+1}) \leqslant \max\left\{\left(\frac{r_1q}{t}\right)^{n-1},\left(\frac{r_1q}{2-r_1q}\right)^{n-1}\right\}\rho(u_1,u_2) \text{ for every } n \in N. \tag{2.2b}$$

Since $r_1q < 1/t$ and $t > 1$ imply

$$\frac{r_1q}{t} < \frac{1}{t^2} < \frac{1}{2t-1},$$

$$\frac{r_1q}{2-r_1q} < \frac{1/t}{2-1/t} = \frac{1}{2t-1},$$

then

$$\max\left\{\frac{r_1q}{t},\frac{r_1q}{2-r_1q}\right\} < \frac{1}{2t-1}.$$

Put

$$\lambda = \frac{1}{2t-1}.$$

Then from (2.2b) we have

$$\rho(u_n,u_{n+1}) \leqslant \lambda^{n-1}\rho(u_1,u_2) \text{ for every } n \in N. \tag{2.2c}$$

Since $1 < t$, then we have $\lambda < 1$. Moreover,

$$t\lambda = \frac{t}{2t-1} < 1.$$

Now, we shall show that $\{u_n\}$ is a Cauchy sequence. From (ii) in Lemma 1.7 and (2.2c) we have

$$\begin{aligned}\rho(u_n,u_{n+p}) &\leqslant t\rho(u_n,u_{n+1}) + t^2\rho(u_{n+1},u_{n+2}) + \cdots + t^{p-1}\rho(u_{n+p-2},u_{n+p-1}) + t^{p-1}\rho(u_{n+p-1},u_{n+p})\\ &\leqslant t(\lambda)^{n-1}\rho(u_1,u_2) + t^2(\lambda)^{n}\rho(u_1,u_2) + \cdots + t^{p-1}(\lambda)^{n+p-3}\rho(u_1,u_2) + t^{p-1}(\lambda)^{n+p-2}\rho(u_1,u_2)\\ &\leqslant t(\lambda)^{n-1}\rho(u_1,u_2)\left[1 + t\lambda + \cdots + (t\lambda)^{p-2} + \frac{1}{t}(t\lambda)^{p-1}\right] \leqslant t(\lambda)^{n-1}\rho(u_1,u_2)[1 + t\lambda + \cdots + (t\lambda)^{p-2} + (t\lambda)^{p-1}].\end{aligned}$$

Since $t\lambda < 1$, then we get

$$\rho(u_n,u_{n+p}) \leqslant t(\lambda)^{n-1}\rho(u_1,u_2)\frac{1-(t\lambda)^p}{1-t\lambda}.$$

Hence it follows,

$$\lim_{n\to\infty}\rho(u_n,u_{n+p}) \leqslant \lim_{n\to\infty} t(\lambda)^{n-1}\rho(u_1,u_2)\frac{1-(t\lambda)^p}{1-t\lambda} = 0$$

for all $p \in \{1,2,3,\ldots\}$. Therefore, we proved that $\{u_n\}$ is a Cauchy sequence in (X,ρ). Now $d(x,y) \leqslant c\rho(x,y)$ implies that $\{u_n\}$ is also a Cauchy sequence in (X,d). By completeness of X, there exist $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now we shall show that

$$D_d(z,Tx) \leqslant \frac{r}{s}\max\{d(z,x), D_d(x,Tx)\} \text{ for all } x \neq z. \tag{2.3}$$

As $\lim_{n\to\infty} u_n = z$, there exist a positive integer $n_0 \in N$ such that $d(z,u_n) \leqslant [1/(3s)]d(z,x)$ for all $n \geqslant n_0$. Since $u_{n+1} \in Tu_n$, then we have

$$\frac{\varphi(r)}{s}D_d(u_n,Tu_n) \leqslant \frac{1}{s}D_d(u_n,Tu_n) \leqslant \frac{1}{s}d(u_n,u_{n+1}) \leqslant d(u_n,z) + d(z,u_{n+1}).$$

Hence for any $n \geqslant n_0$,

$$\frac{\varphi(r)}{s}D_d(u_n,Tu_n) \leqslant \frac{2}{3s}d(z,x).$$

Since

$$\frac{2}{3s}d(z,x) = \frac{1}{s}d(z,x) - \frac{1}{3s}d(z,x) \leqslant \frac{1}{s}d(z,x) - d(u_n,z) \leqslant d(u_n,x),$$

then we have

$$\frac{\varphi(r)}{s}D_d(u_n,Tu_n) \leqslant d(u_n,x).$$

Now from (2.2),

$$\begin{aligned}D_d(u_{n+1},Tx) \leqslant H_d(Tu_n,Tx) &\leqslant \frac{r}{s}\max\left\{d(u_n,x), D_d(u_n,Tu_n), D_d(x,Tx), \frac{D_d(u_n,Tx) + D_d(x,Tu_n)}{2}\right\}\\ &\leqslant \frac{r}{s}\max\left\{d(u_n,x), d(u_n,u_{n+1}), D_d(x,Tx), \frac{D_d(u_n,Tx) + d(x,u_{n+1})}{2}\right\}.\end{aligned}$$

Letting n tends to infinity we get

$$D_d(z,Tx) \leqslant \frac{r}{s}\max\left\{d(z,x), D_d(x,Tx), \frac{D_d(z,Tx) + d(x,z)}{2}\right\}. \tag{2.3a}$$

If we suppose that (2.3) does not hold, that is, that

$$D_d(z,Tx) > \frac{r}{s}\max\{d(z,x), D_d(x,Tx)\}, \tag{2.3b}$$

then from (2.3a),

$$D_d(z,Tx) \leqslant \frac{r}{s}\frac{D_d(z,Tx) + d(x,z)}{2}.$$

Hence we get, as $r < 1 \leqslant s$,

$$D_d(z, Tx) \leqslant \frac{r}{2s - r} d(x, z) < \frac{r}{s} d(x, z) \leqslant \frac{r}{s} \max\{d(z, x), D_d(x, Tx)\}.$$

This is in contradiction with (2.3b). Therefore, from (2.3a) we have that (2.3) holds for all $x \neq z$.

Now we shall prove that $z \in Tz$.

Consider at first the case $0 \leqslant r < 1/2$. Suppose, to the contrary, that $z \notin Tz$. Then, we can choose $a \in Tz$ such that

$$2rd(a, z) < D_d(z, Tz). \tag{2.4}$$

As $a \in Tz$ and $z \notin Tz$ imply $a \neq z$, then from (2.3) with $x = a$,

$$D_d(z, Ta) \leqslant \frac{r}{s} \max\{d(z, a), D_d(a, Ta)\}. \tag{2.5}$$

Since $a \in Tz$ and $\frac{\varphi(r)}{s} D_d(z, Tz) \leqslant D_d(z, Tz) \leqslant d(z, a)$, then from (2.2) we have

$$H_d(Tz, Ta) \leqslant \frac{r}{s} \max\left\{d(z, a), D_d(z, Tz), D_d(a, Ta), \frac{D_d(z, Ta) + D_d(a, Tz)}{2}\right\} \leqslant \frac{r}{s} \max\left\{d(z, a), d(z, a), D_d(a, Ta), \frac{D_d(z, Ta)}{2}\right\}$$
$$\leqslant \frac{r}{s} \max\left\{d(z, a), D_d(a, Ta), \frac{D_d(z, Ta)}{2}\right\}.$$

Hence, and by (2.5), we get

$$H_d(Tz, Ta) \leqslant \frac{r}{s} \max\{d(z, a), D_d(a, Ta)\}. \tag{2.6}$$

Since $a \in Tz$, then $D_d(a, Ta) \leqslant H_d(Tz, Ta)$. Therefore, from (2.6) we have

$$H_d(Tz, Ta) \leqslant \frac{r}{s} \max\{d(z, a), H_d(Tz, Ta)\}.$$

Hence, as $r/s < 1$,

$$H_d(Tz, Ta) \leqslant \frac{r}{s} d(z, a). \tag{2.7}$$

From (2.7),

$$D_d(a, Ta) \leqslant d(z, a).$$

Now by (2.5), (2.7) and (2.4) we have

$$D_d(z, Tz) \leqslant sD_d(z, Ta) + sH_d(Tz, Ta) \leqslant r \max\{d(z, a), D_d(a, Ta)\} + rd(z, a) = rd(z, a) + rd(z, a) < D_d(z, Tz).$$

This is a contradiction. Thus we proved that $z \in Tz$.

Consider now the case $\frac{1}{2} \leqslant r < 1$. We shall prove that

$$H_d(Tx, Tz) \leqslant \frac{r}{s} \max\left\{d(x, z), D_d(x, Tx), D_d(z, Tz), \frac{D_d(x, Tz) + D_d(z, Tx)}{2}\right\} \tag{2.8}$$

for all $x \in X$. If $x = z$ then (2.8) holds trivially. So, assume that $x \neq z$. Then for each $n \in N$, there exists $y_n \in Tx$ such that

$$d(z, y_n) < D_d(z, Tx) + \frac{1}{n} d(x, z).$$

Then by the b-triangle inequality we have

$$D_d(x, Tx) \leqslant d(x, y_n) \leqslant sd(x, z) + sd(z, y_n) < sd(x, z) + sD_d(z, Tx) + \frac{s}{n} d(x, z).$$

Hence, by (2.3),

$$D_d(x, Tx) < sd(x, z) + r \max\{d(z, x), D_d(x, Tx)\} + \frac{s}{n} d(x, z). \tag{2.8a}$$

If $\max\{d(z, x), D_d(x, Tx)\} = d(x, z)$, then from (2.8),

$$D_d(x, Tx) < sd(x, z) + rd(z, x) + \frac{s}{n} d(x, z) < sd(x, z) + rsd(z, x) + \frac{s}{n} d(x, z) < \left[s(1 + r) + \frac{s}{n}\right] d(x, z).$$

This implies

$$\frac{1}{s(1 + r)} D_d(x, Tx) < \left[1 + \frac{1}{(1 + r)n}\right] d(x, z).$$

Hence, as $\varphi(r) = 1 - r$, we have

$$\frac{\varphi(r)}{s}D_d(x,Tx)=\frac{1-r}{s}D_d(x,Tx)\leqslant\frac{1}{s(1+r)}D_d(x,Tx)<\left[1+\frac{1}{(1+r)n}\right]d(x,z).$$

On taking limit as n tends ad infinity we obtain

$$\frac{\varphi(r)}{s}D_d(x,Tx)\leqslant d(x,z).$$

Then from (2.2) with $y=z$ we get (2.8).

If $d(x,z)<D_d(x,Tx)$, then from (2.8a),

$$D_d(x,Tx)\leqslant sd(x,z)+rD_d(x,Tx)+\frac{s}{n}d(x,z).$$

Hence

$$\frac{(1-r)}{s}D_d(x,Tx)\leqslant\left(1+\frac{1}{n}\right)d(x,z).$$

Letting n tends ad infinity we get

$$\frac{(1-r)}{s}D_d(x,Tx)\leqslant d(x,z),$$

that is,

$$\frac{\varphi(r)}{s}D_d(x,Tx)\leqslant d(x,z).$$

Then again from (2.2) with $y=z$ we get (2.8). Thus we proved that (2.8).

From (2.8) with $x=u_n$ we have

$$\begin{aligned}D_d(u_{n+1},Tz)\leqslant H_d(Tu_n,Tz)&\leqslant\frac{r}{s}\max\left\{d(u_n,z),D_d(u_n,Tu_n),D_d(z,Tz),\frac{D_d(u_n,Tz)+D_d(z,Tu_n)}{2}\right\}\\&\leqslant\frac{r}{s}\max\left\{d(u_n,z),\,d(u_n,u_{n+1}),D_d(z,Tz),\frac{D_d(u_n,Tz)+d(z,u_{n+1})}{2}\right\}.\end{aligned}$$

Letting n tends ad infinity we obtain

$$D_d(z,Tz)\leqslant\frac{r}{s}D_d(z,Tz).$$

Hence, as $\frac{r}{s}<1$, we have $D_d(z,Tz)=0$. Since Tz is closed, then $z\in Tz$.

Example 2.1. We know that (l_p,d), where $0<p<1$,

$$l_p=\left\{(x_n)\subset R\left|\sum_{n=1}^{\infty}|x_n|^p<\infty\right.\right\},$$

$d:l_p\times l_p\to R$ is defined by

$$d(x,y)=\left(\sum_{n=1}^{\infty}|x_n-y_n|^p\right)^{1/p},$$

where $x=(x_n),y=(y_n)\in l_p$, is a b-metric space with $s=2^{1/p}$. Let $X\subset l_2$ be defined by $X=\{x_1=(\frac{1}{2^6},0,0,\ldots),x_2=(0,1,0,\ldots),x_3=(0,0,\frac{1}{2^6},\ldots)\}$ and (X,d) be the subspace of l_p. Consider $\rho(x,y)=2d(x,y)$. Both d and ρ are b-metric spaces with $s=2^{\frac{1}{2}}$ and $t=2^{\frac{3}{2}}$. It is noted that $d(x,y)\leqslant c\rho(x,y)$ holds for any $c\geqslant\frac{1}{2}$. Let $\varphi(r)$ be defined as in Theorem 2.1. Define $T:X\to CB(X)$ as follows:

$$Tx=\begin{cases}\{x_1,x_3\} & \text{when } x\neq x_2\\ \{x_1\} & \text{when } x=x_2\end{cases}$$

Let as take $r=\frac{1}{3}$. Then $\varphi(r)=1$ and $0<r<\min\{(2^{\frac{1}{2}})^{-1},(2^{\frac{3}{2}})^{-1}\}<1$. Note that for $x=x_1$ or x_3, we have $D_\rho(x,Tx)=0$. For $x=x_2$

$$D_\rho(x_2,Tx_2)=D_\rho(x_2,\{x_1\})=\rho(x_2,x_1)=\sqrt{\frac{1}{2^{12}}+1}\approx1.000122.$$

So for $x=x_2$ and for all $y\neq x_2$, we have

$$\rho(x_2,y)=\sqrt{\frac{1}{2^{12}}+1}\approx1.000122.$$

This implies that $\frac{\varphi(r)}{t}D_\rho(x_2, Tx_2) \approx \frac{1.000122}{2^{\frac{3}{2}}} \approx 0.3536$. Hence for all $x \neq y$,

$$\frac{\varphi(r)}{t}D_\rho(x, Tx) \leqslant \rho(x, y)$$

holds. For $x \in \{x_1, x_3\}$, and $y \in \{x_1, x_3\}$ we have $H_\rho(Tx, Ty) = 0$. Now

$$H_\rho(Tx_1, Tx_2) = H_\rho(\{x_1, x_3\}, \{x_1\}) = \rho(x_1, x_3) = \sqrt{\frac{1}{2^{11}}} \approx 0.0221.$$

Also, we have

$$H_\rho(Tx_1, Tx_2) = H_\rho(Tx_2, Tx_1) = H_\rho(Tx_2, Tx_3) = H_\rho(Tx_3, Tx_2).$$

As $\rho(x_1, x_2) = \sqrt{\frac{1}{2^{12}} + 1} \approx 1.000122$ and $\frac{r}{t} = \frac{1}{3(2)^{\frac{3}{2}}} \approx 0.11785$, and $\frac{r}{t}\rho(x_1, x_2) = 0.11786$. This implies

$$H_\rho(Tx_1, Tx_2) \leqslant \frac{r}{t}\rho(x_1, x_2).$$

Hence for all $x, y \in X$

$$\frac{\varphi(r)}{t}D_\rho(x, Tx) \leqslant \rho(x, y) \text{ implies } H_\rho(Tx, Ty) \leqslant \frac{r}{t}M_\rho^T(x, y)$$

holds true where

$$M_\rho^T(x, y) = \max\left\{\rho(x, y), D_\rho(x, Tx), D_\rho(y, Ty), \frac{D_\rho(x, Ty) + D_\rho(y, Tx)}{2}\right\}.$$

Similarly, it can be shown that for all $x, y \in X$,

$$\frac{\varphi(r)}{s}D_d(x, Tx) \leqslant d(x, y) \text{ implies } H_d(Tx, Ty) \leqslant \frac{r}{s}M_d^T(x, y).$$

Hence all the conditions of Theorem (2.1) are satisfied. Moreover, x_1 and x_3 are fixed points of T.

From Theorem (2.1) we have the following corollaries.

Corollary 2.2. Let X is a non-empty set, d and ρ be two b-metrics on X with constants $s \geqslant 1$ and $t \geqslant 1$ respectively. Let $\varphi : [0, 1) \to (0, 1]$ be defined as in Theorem 2.1. Suppose that (X, d) is complete b-metric space and there exists $c > 0$ such that $d(x, y) \leqslant c\rho(x, y)$ for all $x, y \in X$. Assume there exist $0 \leqslant r < \min\{\frac{1}{t}, \frac{1}{s}\} < 1$, such that $T : X \to CB(X)$ satisfies

$$\frac{\varphi(r)}{t}D_\rho(x, Tx) \leqslant \rho(x, y) \text{ implies } H_\rho(Tx, Ty) \leqslant \frac{r}{t}\max\{\rho(x, y), D_\rho(x, Tx), D_\rho(y, Ty)\} \tag{2.9}$$

$$\frac{\varphi(r)}{s}D_d(x, Tx) \leqslant d(x, y) \text{ implies } H_d(Tx, Ty) \leqslant \frac{r}{s}\max\{d(x, y), D_d(x, Tx), D_d(y, Ty)\}. \tag{2.10}$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 2.3. *Let X is a nonempty set, d and ρ be two b-metrics on X with constants $s > 1$ and $t > 1$ respectively. Let $\varphi : [0, 1) \to (0, 1]$ be defined as in Theorem 2.1. Suppose that (X, d) is a complete b-metric space and there exists $c > 0$ such that $d(x, y) \leqslant c\rho(x, y)$ for all $x, y \in X$. Assume there exist Let $\alpha, \beta, \gamma, \delta \in [0, 1)$ and $r = \alpha + \beta + \gamma + 2\delta = r < \min\{\frac{1}{t}, \frac{1}{s}\}$ such that $T : X \to CB(X)$ satisfies*

$$\frac{\varphi(r)}{t}D_\rho(x, Tx) \leqslant \rho(x, y) \text{ implies}$$

$$H_\rho(Tx, Ty) \leqslant \frac{1}{t}[\alpha\rho(x, y) + \beta D_\rho(x, Tx) + \gamma D_\rho(y, Ty) + 2\delta(D_\rho(x, Ty) + D_\rho(y, Tx))] \tag{2.11}$$

and

$$\frac{\varphi(r)}{s}D_d(x, Tx) \leqslant d(x, y) \text{ implies}$$

$$H_d(Tx, Ty) \leqslant \frac{1}{s}[\alpha d(x, y) + \beta D_d(x, Tx) + \gamma D_d(y, Ty)] + 2\delta(D_d(x, Ty) + D_d(y, Tx))] \tag{2.12}$$

for all $x, y \in X$. Then T has a fixed point.

For single-valued mappings we have the following result.

Corollary 2.4. *Let X be a nonempty set, d and ρ be two b-metrics on X with constants $s \geqslant 1$ and $t \geqslant 1$ respectively. Let $\varphi(r)$ be defined as in Theorem 2.1. Suppose that (X,d) is a complete b-metric space and there exists $c > 0$ such that $d(x,y) \leqslant c\rho(x,y)$ for all $x,y \in X$. Assume there exist $0 \leqslant r < \min\{\frac{1}{t},\frac{1}{s}\}$ such that $T : X \to X$ satisfies*

$$\frac{\varphi(r)}{t}\rho(x,Tx) \leqslant \rho(x,y) \Rightarrow \rho(Tx,Ty) \leqslant \frac{r}{t}m^T_\rho(x,y), \tag{2.13}$$

and

$$\frac{\varphi(r)}{s}d(x,Tx) \leqslant d(x,y) \Rightarrow d(Tx,Ty) \leqslant \frac{r}{s}m^T_d(x,y), \tag{2.14}$$

for all $x,y \in X$, where

$$m^T_\rho(x,y) = \max\left\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \frac{\rho(x,Ty)+\rho(y,Tx)}{2}\right\},$$
$$m^T_d(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Then T has a fixed point. Now we give an example where Theorem 2.1 of [12] can not be applied, whereas our Theorem 2.1 is applicable.

Example 2.5. Let $X = \{0,1,2\}$ and $d : X \times X \to R$ be defined as

$$d(0,1) = 2,\ \ d(1,2) = \frac{1}{2},\ \ d(0,2) = 1,$$
$$d(0,0) = d(1,1) = d(2,2) = 0,$$
$$d(a,b) = d(b,a)\ \text{ for all }\ a,b \in X.$$

Note that

$$2 = d(0,1) \nleqslant d(0,2) + d(2,1) = 1 + \frac{1}{2}$$

Hence (X,d) is not a metric space. Indeed (X,d) is a b-metric space with $s = \frac{4}{3} > 1$. For simplicity, let a b-metric ρ be defined by $\rho(x,y) = d(x,y)$. Then $t = s$, and we will use the notations $D_d = D_\rho = d$ and $H_d = H_\rho = H$. Let $\varphi(r)$ be given as in Theorem 2.1. Define $T : X \to CB(X)$ as follows:

$$Tx = \begin{cases} \{0,2\} & \text{when } x \neq 1 \\ \{0\} & \text{when } x = 1 \end{cases}$$

If we take $r = \frac{7}{9} < \frac{1}{s}$, then $\varphi(r) = \frac{2}{9}$. Note that for $x = 0$ or 2, we have $d(x,Tx) = 0$. For $x = 1, d(1,T1) = 2$. Since

$$\max\{d(x,Tx) : x \in X\} = 2;\ \ \min\{d(x,y) : x,y \in X\} = \frac{1}{2}$$

and as $\frac{\varphi(r)}{s} = \frac{1}{6}$, we have that

$$\frac{\varphi(r)}{s}d(x,Tx) \leqslant d(x,y)$$

holds for all $x,y \in X$. For $x \in \{0,2\}$, and $y \in \{0,2\}$ we have $H(Tx,Ty) = 0$. Now, as

$$H(T1,T2) = H(T1,T0) = H(\{0\},\{0,2\}) = 1,$$

we have

$$H(T1,T2) < \frac{7}{8}\cdot 2 = \frac{r}{s}d(1,T1) = \frac{r}{s}M^T(1,2),$$
$$H(T1,T0) < \frac{7}{8}\cdot 2 = \frac{r}{s}d(1,0) = \frac{r}{s}M^T(0,1).$$

Therefore, for all $x,y \in X$

$$\frac{\varphi(r)}{s}d(x,Tx) \leqslant d(x,y)\ \text{ implies }\ H(Tx,Ty) \leqslant \frac{r}{s}M^T(x,y).$$

So all the conditions of Theorem 2.1 are satisfied. Moreover, 0 and 2 are fixed points of T.

On the other hand, if we take usual metric d on X, then again

$$\varphi(r)d(x,Tx) \leqslant d(x,y)$$

holds for all $x,y \in X$. Now for $x = 1$ and $y = 2$ we have

$$H(T1, T2) = H(\{0\}, \{0, 2\}) = 2.$$

Also,

$$d(1,2) = 1, \;\; d(1,T1) = 1, \;\; d(2,T2) = 0, \;\; d(1,T2) = 1, \;\; d(2,T1) = 2,$$

implies that $M(1,2) = \frac{3}{2}$, where

$$M^T(x,y) = \max\left\{d(1,2), d(1,T1), d(2,T2), \frac{d(1,T2) + d(2,T1)}{2}\right\} = \max\left\{1, 1, 0, \frac{1+2}{2}\right\} = \frac{3}{2}.$$

Thus

$$H(T1, T2) = 2 \nleqslant \frac{r}{1} \cdot \frac{3}{2} = rM(1,2)$$

for any $r \in [0, 1)$. So Theorem 2.1 of Djorić and Lazović in [12] is not applicable in this case.

Remark. If we take $\rho = d$ and $s = t = 1$ in Theorem 2.1, then it reduces to the Theorem 2.1 of [12]. Moreover, Theorem 2.1 provides the answer to the Question 1 posed in [25] in spaces endowed with b-metrics.

Corollary 2.2 is the generalization of Corollary 2.1 in [12] and therefore it further generalizes Theorem 2.2 in [15] and Kannan fixed point theorem in [14].

Corollary 2.4 extends Corollary 2.3 in [12], which in turn becomes the generalization of results in [14] and Theorem 3.1 in [13].

3. Applications

In this section we assume U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$. Let $\mathbb{R}$ be the field of real numbers, $\tau : W \times D \to W$, $g : W \times D \to \mathbb{R}$, and $G : W \times D \times \mathbb{R} \to \mathbb{R}$. Consider here W and D as the state and decision spaces respectively. It is known that the problem of dynamic programming related to multistage process reduces to the problem of solving the functional equation:

$$p(x) := \sup_{y \in D}\{g(x,y) + G(x, y, p(\tau(x,y)))\}, x \in W. \tag{3.1}$$

For the detailed background of the problem (see [2,3,5,1,20,23]). In this section, we study the existence of solution of the functional Eq. (3.1).

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W} |hx|$. Then $(B(W), \|\cdot\|)$ is a Banach space endowed with the metric

$$\rho_B(h,k) = \sup_{x \in W} |hx - kx|,$$

where $h, k \in B(W)$.

Suppose that the following conditions hold:

$(DT - 1)$ G, g are bounded.

$(DT - 2)$ Let $d_B = k\rho_B$ for $k > 0$, then $d_B(h,k) \leqslant c\rho_B(h,k)$ is satisfied for any $c \geqslant k > 0$. d_B and d_B are b-metrics on $B(W)$ with any constants $s > 1$ and $t > 1$ respectively.

$(DT - 3)$ $\varphi(r)$ be defined as in Section (1) and for $h, k \in B(W)$ and $x, z \in W$. Let T be defined as

$$T(hx) := \sup_{y \in D}\{g(x,y) + G(x, y, h(\tau(x,y)))\}. \tag{3.2}$$

There exist $0 \leqslant r < \min\{\frac{1}{t}, \frac{1}{s}\}$ such that

$$\frac{\varphi(r)}{t}\rho_B(h, Th) \leqslant \rho_B(h,k) \Rightarrow |G(x,y,hz) - G(x,y,kz)| \leqslant \frac{r}{t} m^T_{\rho_B}(hz, kz)$$

and

$$\frac{\varphi(r)}{s} d_B(h, Th) \leqslant d_B(h,k) \Rightarrow |G(x,y,hz) - G(x,y,kz)| \leqslant \frac{r}{s} m^T_{d_B}(hz, kz),$$

where

$$m^T_{\rho_B}(hz, kz) = \max\left\{\rho_B(hz,kz), \rho_B(hz, Thz), \rho_B(kz, Tkz), \frac{\rho_B(hz, Tkz) + \rho_B(kz, Thz)}{2}\right\} \tag{3.3}$$

and, similarly,

$$m^T_{d_B}(hz, kz) = \max\left\{d_B(hz,kz), d_B(hz, Thz), d_B(kz, Tkz), \frac{d_B(hz, Tkz) + d_B(kz, Thz)}{2}\right\}. \tag{3.3a}$$

Theorem 3.1. *Assume that the conditions* $(DT-1)-(DT-3)$ *are satisfied. Then the functional Eq.* (3.1) *has a unique bounded solution.*

Proof. Notice that $(B(W),\rho_B)$ is a complete metric space, where ρ_B is the metric induced by the supremum norm on $B(W)$. T is a self map of $B(W)$. Let λ be an arbitrary positive number and $h_1,h_2\in B(W)$. Pick $x\in W$ and choose $y_1,y_2\in D$ such that

$$Th_1x<g(x,y_1)+G(x,y_1,h_1(\tau(x,y_1)))+\lambda, \tag{3.4}$$

$$Th_2x<g(x,y_2)+G(x,y_2,h_2(\tau(x,y_2)))+\lambda. \tag{3.5}$$

Further

$$Th_1x\geqslant g(x,y_2)+G(x,y_2,h_1(\tau(x,y_2)))$$

and

$$Th_2x\geqslant g(x,y_1)+G(x,y_1,h_2(\tau(x,y_1))).$$

Then from (3.4) and (3.7) implies

$$\begin{aligned}Th_1x-Th_2x&<G(x,y_1,h_1(\tau(x,y_1)))-G(x,y_1,h_2(\tau(x,y_2)))+\lambda\leqslant|G(x,y_1,h_1(\tau(x,y_1)))-G(x,y_1,h_2(\tau(x,y_2)))|+\lambda\\&\leqslant\frac{r}{t}m^T_{\rho_B}(h_1x,h_2x)+\lambda\end{aligned} \tag{3.8}$$

and from (3.5) and (3.6),

$$Th_2x-Th_1x\leqslant\frac{r}{t}m^T_{\rho_B}(h_1x,h_2x)+\lambda. \tag{3.9}$$

So from (3.8) and (3.9), we have

$$|Th_1x-Th_2x|\leqslant\frac{r}{t}m^T_{\rho_B}(h_1x,h_2x)+\lambda. \tag{3.10}$$

That is

$$\rho_B(Th_1,Th_2)\leqslant\frac{r}{t}m^T_{\rho_B}(h_1x,h_2x).$$

Since the above inequality holds for any $x\in W$, and $\lambda>0$ is taken arbitrary, then

$$\frac{\varphi(r)}{t}\rho_B(hz,Thz)\leqslant\rho_B(hz,kz)\Rightarrow\rho_B(Th_1,Th_2)\leqslant\frac{r}{t}m^T_{\rho_B}(h_1x,h_2x).$$

Similarly we can show, by similar lines as above, that

$$\frac{\varphi(r)}{s}d_B(hz,Thz)\leqslant d_B(hz,kz)\Rightarrow d_B(Th_1,Th_2)\leqslant\frac{r}{s}m^T_{d_B}(h_1x,h_2x).$$

Therefore, the mapping T satisfies all conditions of Corollary 2.4. Thus, functional Eq. (3.1) has a bounded solution.

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